WARSAW UNIVERSITY OF TECHNOLOGY
DEVELOPMENT PROGRAMME

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Image Restoration and Reconstruction

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What is the Problem?

- Fundamental signal/imaging equation is

$$s = \mathcal{L}f + n$$

s - signal/image
f - information
n - noise
$\mathcal{L}$ - linear operator

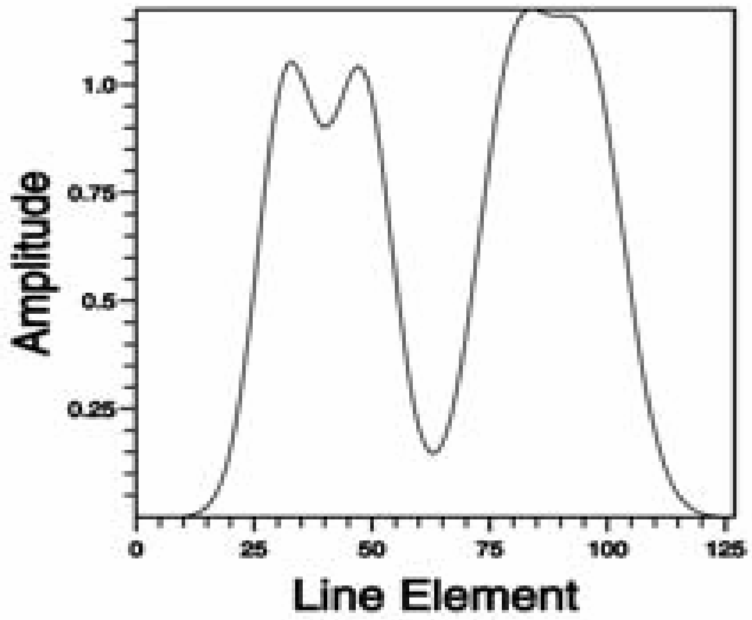
- Given the signal/image, retrieve the information given knowledge of

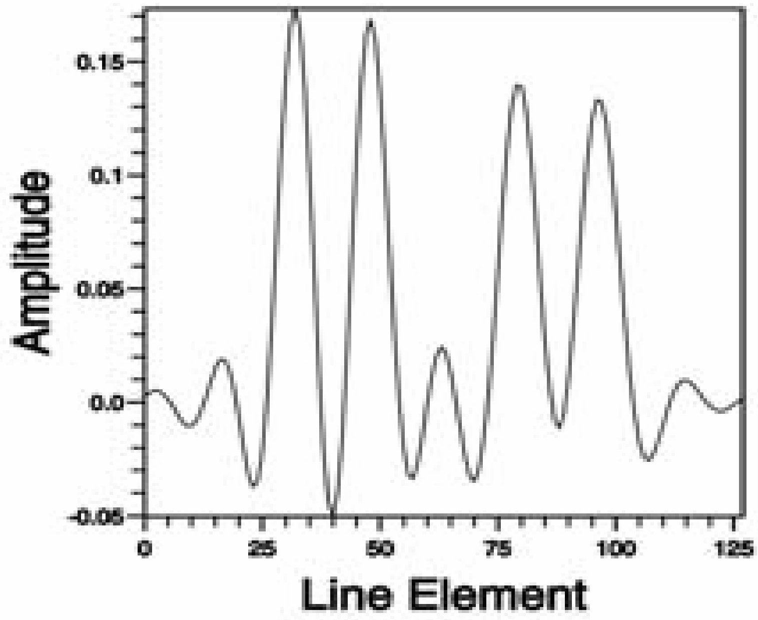
  - linear operator

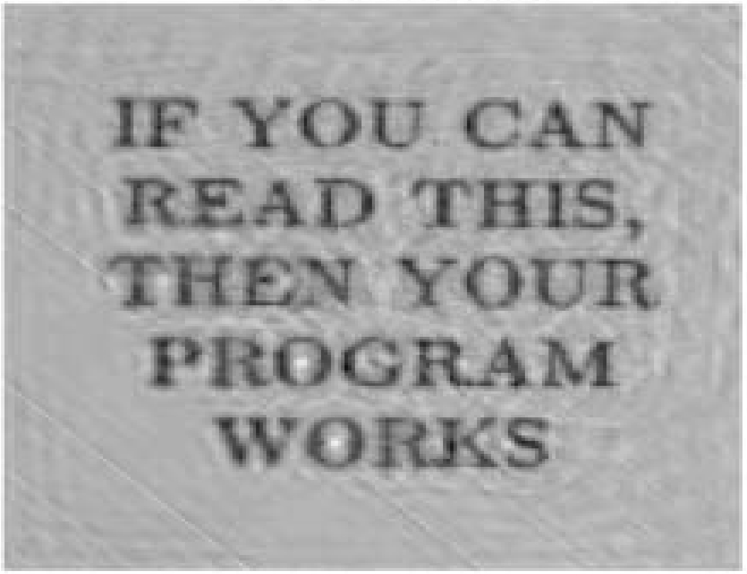
  - noise statistics

- This is an ***inverse problem***

Representative Examples

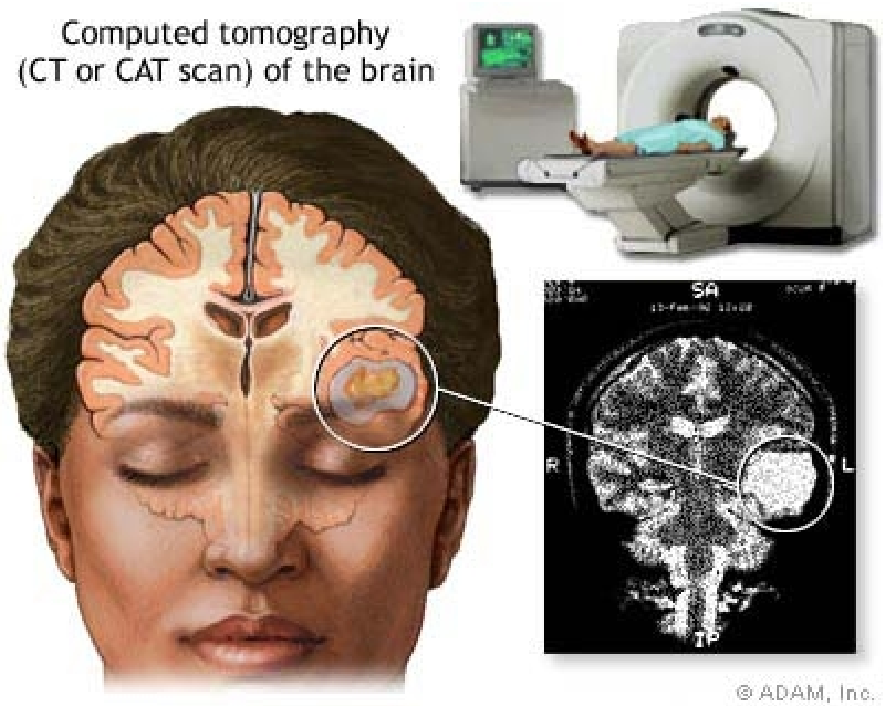




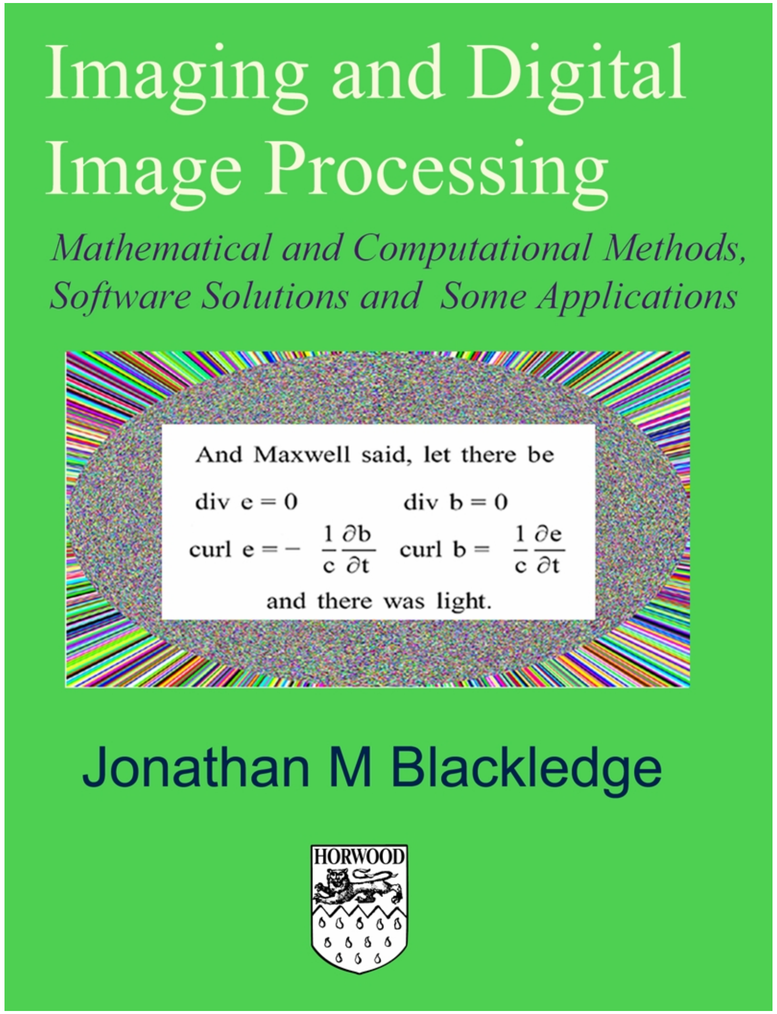


Example Applications

- Medical imaging – ***CT and MRI***
- X-ray Crystallography – ***Phase retrieval***
- Astronomy
- Microscopy
- Seismic prospecting
- Projection Tomography
- Diffraction Tomography
- Forensic image analysis
- Sound/source separation



Principal Publication



http://eleceng.dit.ie/papers/103.pdf

Contents of Presentation I

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- The inverse Helmholtz scattering problem
- Coherence .v. Incoherence
- The Importance of Phase
- The Phase Retrieval Problem
- Phase Imaging
- Deconvolution
- ***Case Study: The Wiener Filter***
- Reconstruction from Bandlimited Data
- Summary
- Q & A + Interval (10 Minutes)

Contents of Presentation II

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- Scattering from Random Media
- Diffusion Based Model
- Inverse Diffusion Imaging
- ***Case Study: Fractional Diffusion Imaging***
- Scattering from Tenuous Random Media
- Example results
- Open Problems
- Summary
- Q & A

Inverse Helmholtz Scattering: *Conventional Solution Method*

$$(\nabla^2 + k^2)u(\mathbf{r}, k) = -k^2\gamma(\mathbf{r})u(\mathbf{r}, k)$$

Model

$$\downarrow$$

$$\text{Green's function transformation}$$

$$\downarrow$$

$$u(\mathbf{r}, k) = u_i(\mathbf{r}, k) + k^2 g(r, k) \otimes_3 \gamma(\mathbf{r})u(\mathbf{r}, k)$$

Scattering equation

$$\downarrow$$

$$\text{Born approximation}$$

$$\downarrow$$

$$u_s(\mathbf{r}, k) = k^2 g(r, k) \otimes_3 \gamma(\mathbf{r})u_i(\mathbf{r}, k)$$

Single scattering equation

Farfield Approximation

$$u_s(\mathbf{r},k) = k^2 g(r,k) \otimes_3 \gamma(\mathbf{r}) u_i(\mathbf{r},k)$$

$$\downarrow$$

$$\text{Farfield} \quad \text{approximation}$$

$$\downarrow$$

$$u_s(k\hat{\mathbf{n}}) \sim \mathcal{F}[\gamma(\mathbf{r})]$$

Fourier transform

$$\downarrow$$

$$\text{Inverse} \quad \text{scattering} \quad \text{solution}$$

$$\downarrow$$

$$\gamma(\mathbf{r}) \sim \mathcal{F}^{-1}[u_s(k\hat{\mathbf{n}})]$$

Inverse Fourier transform

Imaging Equation

$$u_s(k\hat{\mathbf{n}}) \sim \mathcal{F}[\gamma(\mathbf{r})]$$

$$\downarrow$$

$$\text{Detected} \quad \text{signal}$$

$$\downarrow$$

$$S(k\hat{\mathbf{n}}) \sim P(k\hat{\mathbf{n}})\mathcal{F}[\gamma(\mathbf{r})]$$

$$\downarrow$$

$$\text{Convolution} \quad \text{Theorem}$$

$$\downarrow$$

$$s(\mathbf{r}) \sim p(\mathbf{r}) \otimes_3 \gamma(\mathbf{r})$$

- The detected signal gives a limited spectrum of the scattering function characterised by a stationary ***Point Spread Function (PSF)***

- The result is based on the assumption that multiple scattering is negligible

- All non-ideal aspects of this equation including physical effects, signal detection noise etc. are compounded in an additive noise function so that the imaging equation becomes

$$s(\mathbf{r}) = p(\mathbf{r}) \otimes_3 \gamma(\mathbf{r}) + n(\mathbf{r})$$

Attributes of an Image

Image = (PSF) convolved (Object Function) + Noise

- **Resolution:** determined by the spread of the PSF
- **Distortion:** determined by accuracy of model for PSF
- **Fuzziness:** determined by accuracy of model for object function
- **Noise:** determined by accuracy of convolution model for the image

Coherence .v. Incoherence

- In coherent imaging measures of both the ***amplitude and phase*** are detected

$$u_s(k\hat{\mathbf{n}}) \sim \mathcal{F}[\gamma(\mathbf{r})]$$

- In incoherent imaging only a measure of the ***amplitude (intensity)*** is detected

$$\mid u_s(k\hat{\mathbf{n}}) \mid^2 \sim \mid \mathcal{F}[\gamma(\mathbf{r})] \mid^2$$

Image Types

- An image is usually a measure of the intensity of a scattered field

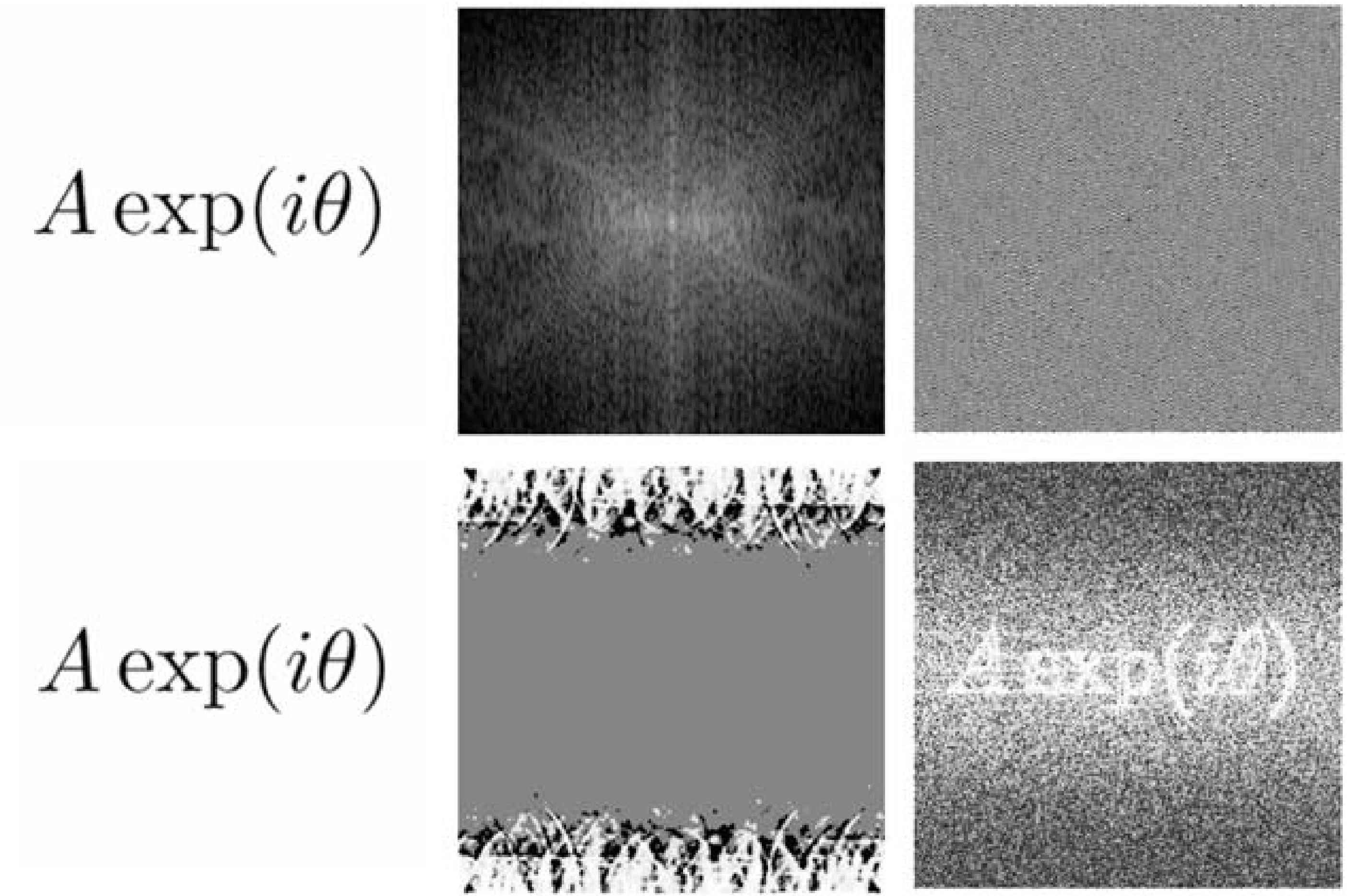
- The model for a coherent image is

$$I_{\text{coherent}} = | p \otimes\otimes f + n |^2$$

- The model for an incoherent image is

$$I_{\text{incoherent}} = | p |^2 \otimes\otimes | f |^2 + | n |^2$$

The Importance of Phase



Original image (top-left), amplitude spectrum displayed using a logarithmic scale (top-centre), phase modulus spectrum (top-right), reconstruction using both the amplitude and phase spectra (bottom-left), ***amplitude only reconstruction*** (bottom-centre) and a ***phase only reconstruction*** (bottom-right).

Phase Retrieval Problem

$$\text{Given } | F(k_x, k_y) |, \text{ find } f(x, y)$$

$$F(k_x, k_y) = \int_{-X/2}^{X/2} \int_{-Y/2}^{Y/2} f(x, y) \exp(-ik_x x) \exp(-ik_y y) dx dy$$

- Important in applications where only the intensity of a ***Born scattering wavefield*** can be measured

- This is always the case when the radiation is high frequency, e.g. ***X-ray Crystallography***

The Fienup Algorithm

$$f(x,y) \quad \rightarrow \quad \hat{F}_2[f(x,y)] \quad \rightarrow \quad |\,F(k_x,k_y)\,|\exp[i\theta(k_x,k_y)]$$

$\uparrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

Conform to region of support

Iterative cycle

Satisfy amplitude constraint

$\uparrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$$f'(x,y) \quad \leftarrow \quad \mathrm{Re}\{\hat{F}_2^{-1}[F'(k_x,k_y)]\} \quad \leftarrow \quad F'(k_x,k_y) = A(k_x,k_y)\exp[i\theta(k_x,k_y)]$$

Phase Imaging

$$s(x,y) = f(x,y) + iq(x,y)$$

$$= A(x,y)\exp[i\theta(x,y) \pm 2\pi i n], \quad n = 0, 1, 2, \ldots$$

$$\theta(x,y) = \mathrm{Im}[\ln s(x,y)] \mp 2\pi n$$

$$\nabla\theta(x,y) = \mathrm{Im}[\nabla \ln s(x,y)]$$

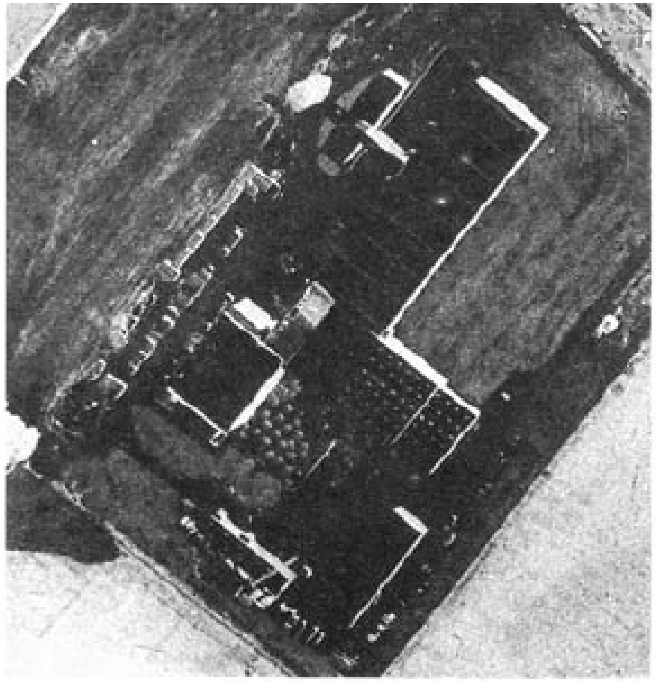
Image generated of the ***Instantaneous Frequency***

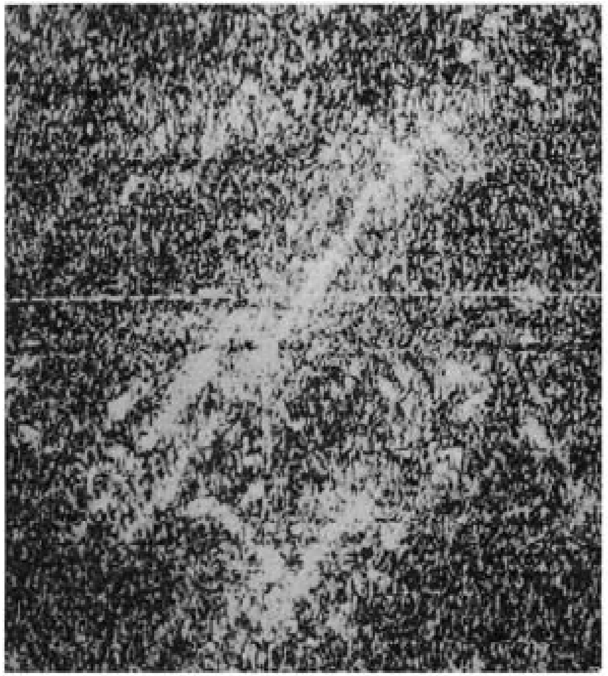
$$|\ \nabla\theta(x,y)\ |$$

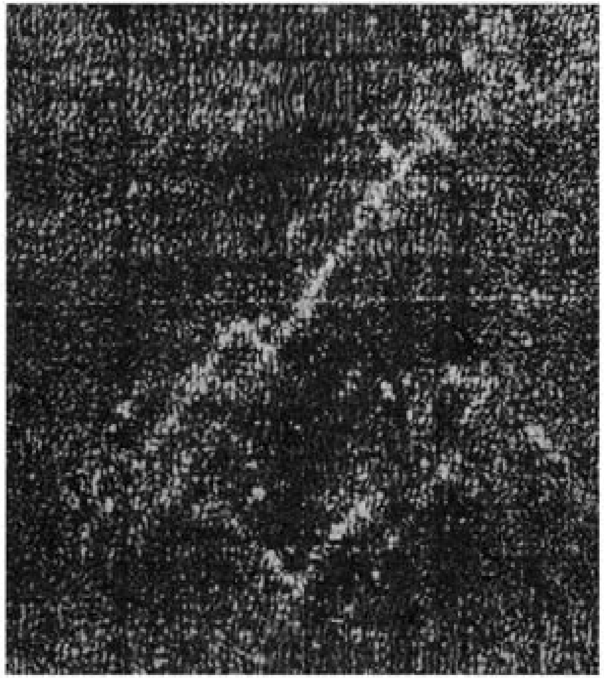
SAR FM Imaging

Optical image

Synthetic Aperture Radar Images



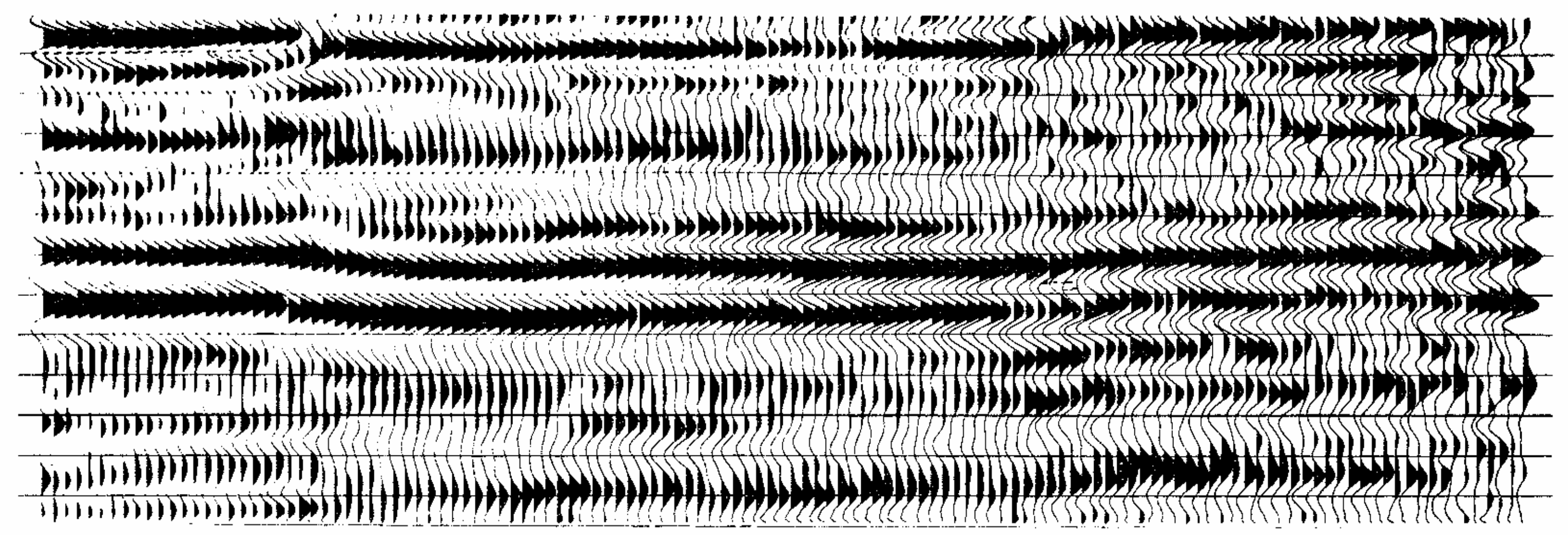




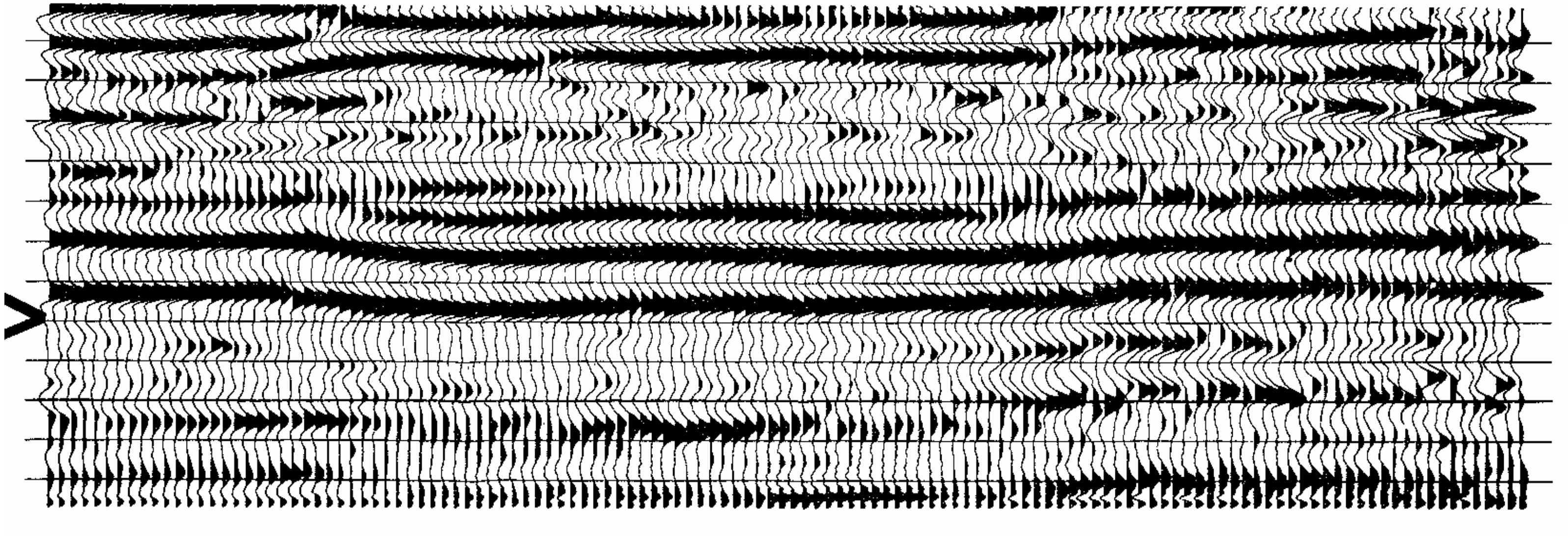
$A(x, y)$

$|\nabla\theta(x, y)|$

Seismic FM Imaging



$f(x, y)$



$\nabla\theta(x, y)$

Deconvolution

- Problem: Given that

$$s_{ij} = p_{ij} \otimes \otimes f_{ij} + n_{ij}$$

develop an algorithm to obtain an estimate of the ***object function***

- Assumes a ***stationary process*** for image formation

- Many solutions to this problem which depend on:

- the ***PSF***
- the ***criterion*** used
- characteristics of the ***noise***

Case Study:

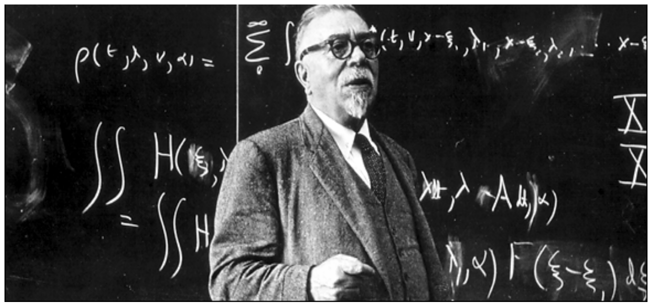
The Wiener Filter

Let s_i be a digital signal consisting of N real numbers $i = 0, 1, 2, \ldots, N\text{-}1$ that has been generated via the time invariant linear process (where p_i – the IRF – is known)

$$s_i = \sum_j p_{i-j} f_j + n_i \qquad \sum_j \equiv \sum_{j=0}^{N-1}$$

find an estimate for f_i of the form

$$\hat{f}_i = \sum_j q_j s_{i-j}$$



Solution

- Consider the least squares error

$$e = \|f_i - \hat{f}_i\|_2^2 \equiv \sum_{i=0}^{N-1} (f_i - \hat{f}_i)^2$$

- e is a minimum when

$$\frac{\partial}{\partial q_k} e(q_j) = 0 \quad \forall k$$

i.e. when

$$\sum_{i=0}^{N-1} \left(f_i - \sum_j q_j s_{i-j} \right) s_{i-k} = 0$$

Solution (continued)

- Using the convolution and correlation theorems

$$F_i S_i^* = Q_i S_i S_i^*$$

$$Q_i = \frac{S_i^* F_i}{| S_i |^2}$$

- Since $S_i = P_i F_i + N_i$

$$Q_i = \frac{P_i^* | F_i |^2 + N_i^* F_i}{| P_i |^2 | F_i |^2 + | N_i |^2 + P_i F_i N_i^* + N_i P_i^* F_i^*}$$

Signal Independent Noise

- We can not compute Q_i because we do not know F_i or N_i

- However, we can expect that the information content of the signal f_i will not correlate with the noise n_i which means that

$$\sum_j n_{j-i} f_j = 0 \quad \text{and} \quad \sum_j f_{j-i} n_j = 0$$

$$N_i^* F_i = 0 \quad \text{and} \quad F_i^* N_i = 0$$

Signal Independent Noise Solution

Given that the noise is signal independent

$$Q_i = \frac{P_i^* \mid F_i \mid^2}{\mid P_i \mid^2 \mid F_i \mid^2 + \mid N_i \mid^2}$$

and

$$Q_i = \frac{P_i^*}{\mid P_i \mid^2 + \frac{|N_i|^2}{|F_i|^2}}$$

Computing the Signal-to-Noise-Ratio

- **Problem:** How can we find $\mid F_i \mid^2 / \mid N_i \mid^2$?
- Suppose we have a linear stationary process whereby we can record a signal twice at different times. Then

$$s_i = p_i \otimes f_i + n_i$$

$$s'_i = p_i \otimes f_i + n'_i$$

$$n_i \odot n'_i = 0, \quad f_i \odot n_i = 0, \quad n_i \odot f_i = 0,$$

$$f_i \odot n'_i = 0, \quad n'_i \odot f_i = 0.$$

Autocorrelation and Cross-correlation Functions

- ***Auto-correlating:*** $c_i = s_i \odot s_i$

$$C_i = S_i S_i^* = | P_i |^2 | F_i |^2 + | N_i |^2$$

- ***Cross-correlating:*** $c_i' = s_i \odot s_i'$

$$C_i' = | P_i |^2 | F_i |^2$$

$$\frac{| N_i |^2}{| F_i |^2} = \left(\frac{C_i}{C_i'} - 1 \right) | P_i |^2$$

Practical Implementation

- Given that the signal-to-noise power ratio is not usually known, i.e. $|F_i|^2 / |N_i|^2$ we approximate the filter as

$$Q_i \sim \frac{P_i^*}{|P_i|^2 + \Gamma} \qquad \Gamma \sim \frac{1}{(\mathrm{SNR})^2}$$

- The value of the SNR (Signal-to-Noise-Ratio) becomes a user defined constant

FFT Algorithm for the Wiener Filter

```
snr=snr*snr
constant=1/snr

for i=1, 2, ..., n; do:
    sr(i)=signal(i)
    si(i)=0.
    pr(i)=IRF(i)
    pi(i)=0.
enddo
        forward_fft(sr,si)
        forward_fft(pr,pi)
```

FFT Algorithm for the Wiener Filter (continued)

```
for i=1, 2, ..., n; do:
    denominator=pr(i)*pr(i)+pi(i)*pi(i)+constant

                fr(i)=pr(i)*sr(i)+pi(i)*si(i)
                fi(i)=pr(i)*si(i)-pi(i)*sr(i)
                fr(i)=fr(i)/denominator
                fi(i)=fi(i)/denominator
enddo
inverse_fft(fr,fi)

 for i=1, 2, ..., n; do:
     hatf(i)=fr(i)
 enddo
```

Signal Restoration using the *Wiener Filter*

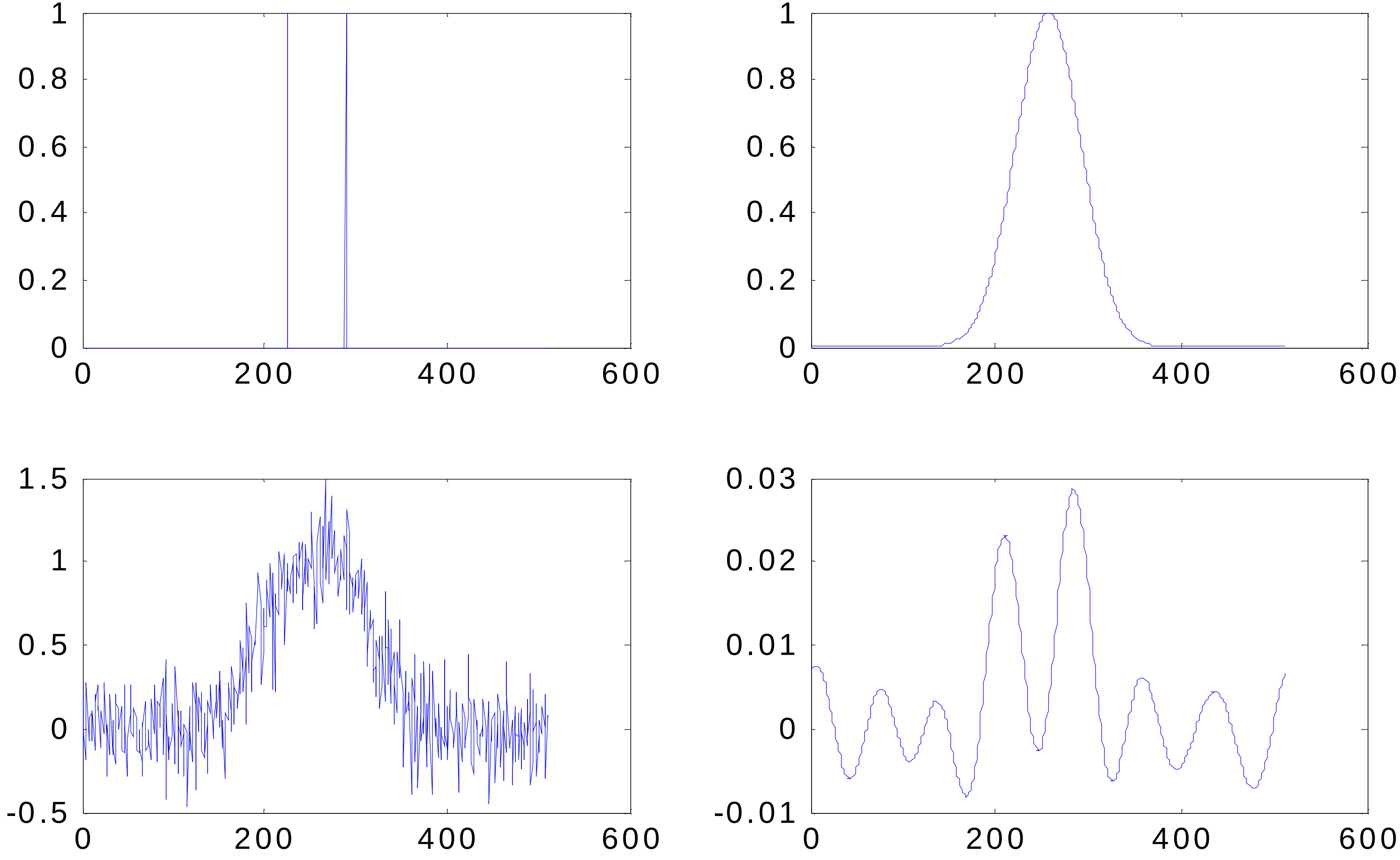
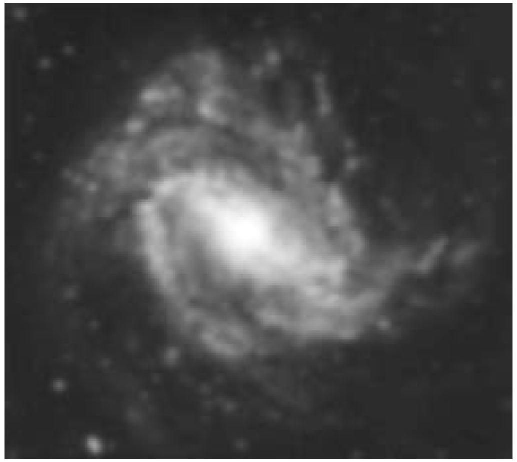


Image Restoration using the Wiener Filter



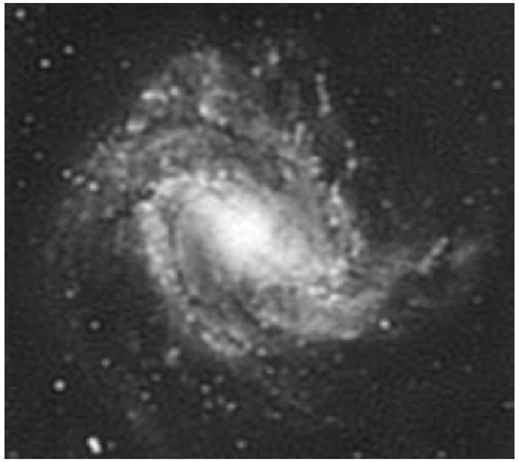
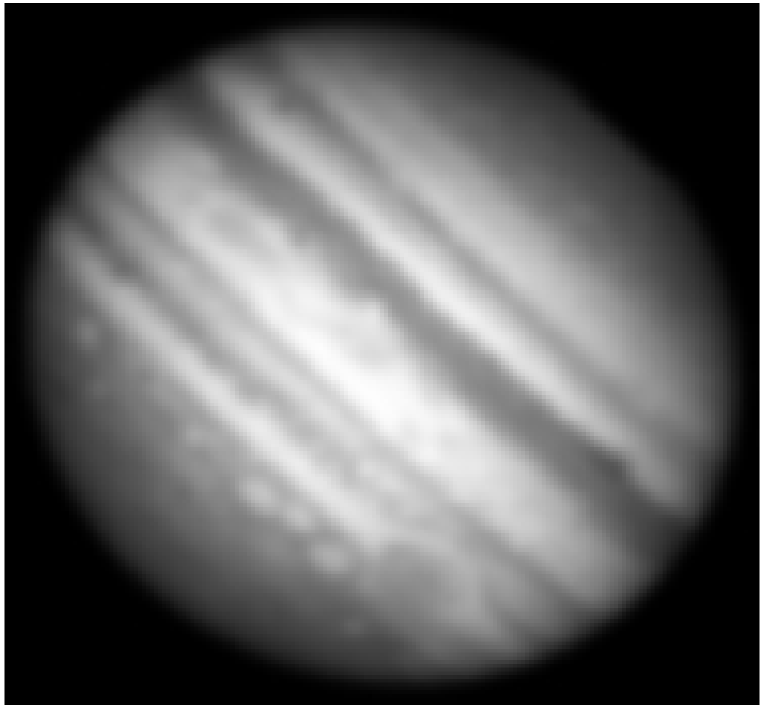
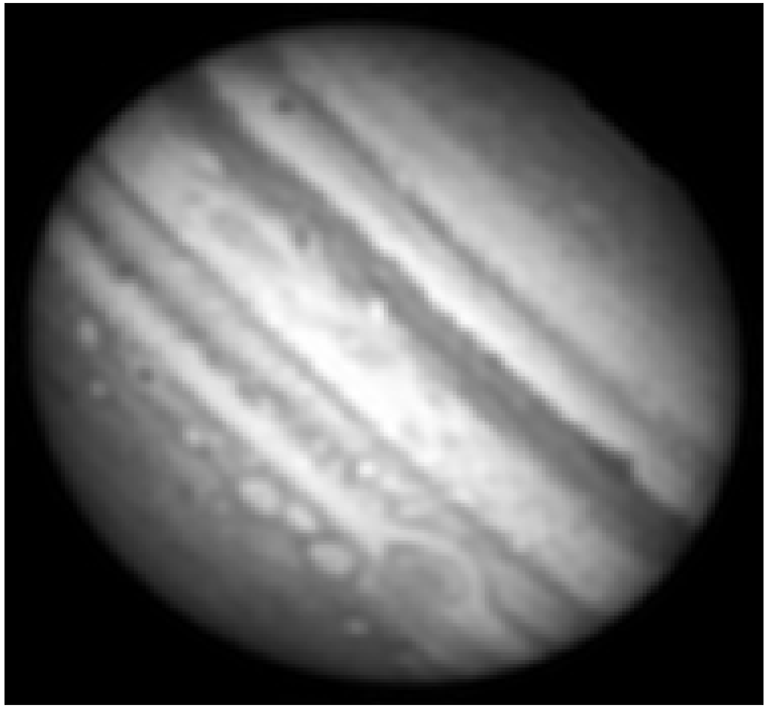


Image restoration using the Wiener filter.

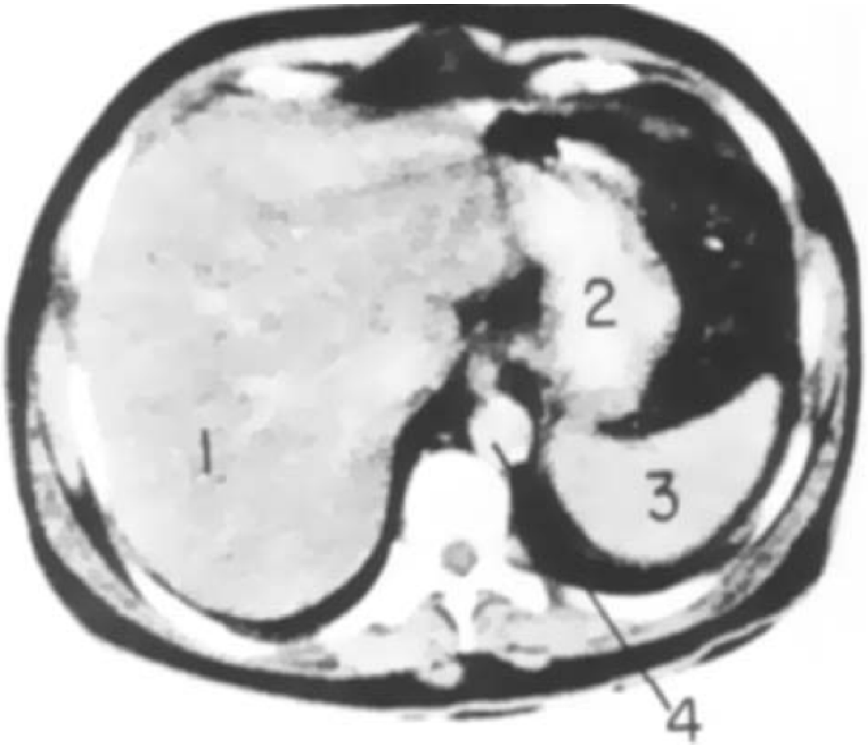
Original image (left), Gaussian PSF (center) and restoration after application of the Wiener filter (right) using a standard deviation of 3 pixels (for the Gaussian PSF) and an SNR=1.

A Further Example





Inverse Filtering in CT



- ***Back-projection function*** is given by

$$B(x,y) = \frac{1}{\sqrt{x^2+y^2}} \otimes \otimes f(x,y)$$

- Reconstruction is given by

$$f(x,y) =$$
$$\hat{F}_2^{-1}[\sqrt{k_x^2+k_y^2}\widetilde{B}(k_x,k_y)]$$

An X-ray tomogram of a normal abdomen showing the Liver (1), Stomach (2), Spleen (3) and Aorta (4).

Filter Types

Name of Filter	Filter	Condition(s)
Inverse	$Q_{ij} = P^*_{ij} / \mid P_{ij} \mid^2$	Min $\|n_{ij}\|$
Wiener	$Q_{ij} = \frac{P^*_{ij}}{\lvert P_{ij}\rvert^2 + \lvert F_{ij}\rvert^2 / \lvert N_{ij}\rvert^2}$	Min $\|f_{ij} - q_{ij} \otimes \otimes s_{ij}\|^2$; $N^*_{ij} F_{ij} = 0, \quad F^*_{ij} N_{ij} = 0$
PSE	$Q_{ij} = \left(\frac{1}{\lvert P_{ij}\rvert^2 + \lvert F_{ij}\rvert^2 / \lvert N_{ij}\rvert^2} \right)^{\frac{1}{2}}$	$\mid F_{ij} \mid^2 = \mid Q_{ij} S_{ij} \mid^2$; $N^*_{ij} F_{ij} = 0, \quad F^*_{ij} N_{ij} = 0$
Matched	$Q_{ij} = P^*_{ij} / \mid N_{ij} \mid^2$	Max $\frac{\lvert \sum_i \sum_j Q_{ij} P_{ij} \rvert^2}{\sum_i \sum_j \lvert N_{ij}\rvert^2 \lvert Q_{ij}\rvert^2}$
Max Entropy	$Q_{ij} = \frac{P^*_{ij}}{\lvert P_{ij}\rvert^2 + 1/\lambda}$	Max $-\sum_i \sum_j f_{ij} \ln f_{ij}$
Constrained	$Q_{ij} = \frac{P^*_{ij}}{\lvert P_{ij}\rvert^2 + \gamma \lvert G_{ij}\rvert^2}$	Min $\|g_{ij} \otimes \otimes f_{ij}\|^2$

Image Reconstruction from Band-limited Data

Problem: Given

$$f_{BL}(x,y) = \sum_n \sum_m F_{nm} e^{i(k_n x + k_m y)}$$

$$F_{nm} = \int_{-X}^{X} \int_{-Y}^{Y} f(x,y) e^{-i(k_n x + k_m y)} dx dy$$

compute an estimate for $f(x,y)$

Gerchberg-Papoulis Method

$$\hat{f}(x,y) = \sum_n \sum_m A_{nm} e^{i(k_n x + k_m y)}$$

$$E = \int_{-X}^{X} \int_{-Y}^{Y} \mid f(x,y) - \hat{f}(x,y) \mid^2 dxdy$$

$$F_{pq} = 4XY \sum_n \sum_m A_{nm} \, \mathrm{sinc}[(k_p - k_n)X] \, \mathrm{sinc}[(k_q - k_m)Y]$$

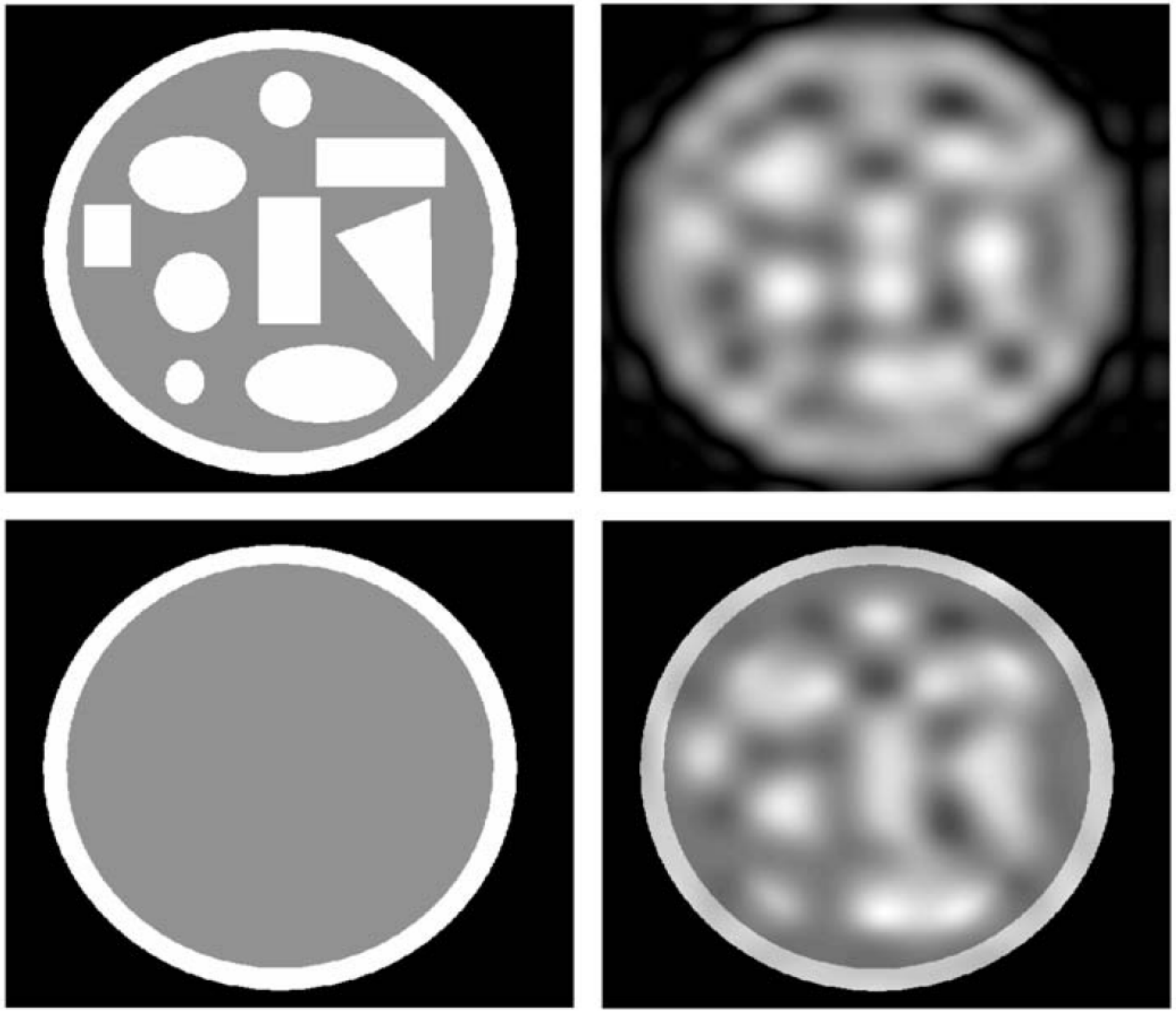
Weighting Function Method

$$\hat{f}(x,y) = w(x,y)\sum_n \sum_m A_{nm} e^{i(k_n x + k_m y)}$$

$$E = \int_{-X}^{X}\int_{-Y}^{Y} \mid f(x,y) - \hat{f}(x,y) \mid^2 \frac{1}{w(x,y)} dx dy.$$

$$\hat{f}(x,y) = \frac{w(x,y)}{w_{BL}(x,y)} f_{BL}(x,y)$$

Example Reconstruction



Reconstruction (bottom-right) of a test object (top-left) function from band-limited data (top-right) using prior information (bottom-left).

Summary

- Most image restoration/reconstruction algorithms are based on a ***stationary convolution model with additive noise***

- The model assumes that the scattered field is detected/measured in the ***far field*** and is the result of ***single scattering processes***

- Image coherence is determined by whether a measure of the phase information can be obtained

In the Following Lecture…

- We shall consider diffusion based models for the scattering of waves from random media

- Develop inverse solutions for ***diffusion imaging***

- Develop inverse solution for ***fractional diffusion imaging***

Questions

+

Interval (10 Minutes)

Part II: Contents

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- Scattering from Random Media
- Diffusion Based Model
- Inverse Diffusion Imaging
- ***Case Study: Fractional Diffusion Imaging***
- Scattering from Tenuous Random Media
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Scattering From Random Media

Basic Problem: Given that

$$\left(\nabla^2 - \frac{1}{c^2(\mathbf{r})}\frac{\partial^2}{\partial t^2}\right)u(\mathbf{r},t) = 0$$

compute $I = | u |^2$ for known $\Pr[c(\mathbf{r})]$

Weak Scattering Model

Scattered wave amplitude in the far field is determined by the Fourier transform

$$A(\hat{\mathbf{N}}, k) = k^2 \int_V \exp(-ik\hat{\mathbf{N}} \cdot \mathbf{r})\gamma(\mathbf{r})d^3\mathbf{r}$$

$$\frac{1}{c^2} = \frac{1}{c_0^2}(1+\gamma) \qquad \hat{\mathbf{N}} = \hat{\mathbf{n}}_s - \hat{\mathbf{n}}_i$$

Intensity of Scattered Field

- Intensity determined by Fourier transform of the autocorrelation function

$$I(\hat{\mathbf{N}}, k) = k^4 \int_V \exp(-ik\hat{\mathbf{N}} \cdot \mathbf{r})\Gamma(\mathbf{r})d^3\mathbf{r}$$

$$\Gamma(\mathbf{r}) = \int_V \gamma(\mathbf{r}')\gamma^*(\mathbf{r}' + \mathbf{r})d^3\mathbf{r}'$$

- Requires a model for the ***autocorrelation function*** that best characterises the random medium, i.e. the ***Power Spectral Density Function***

Examples of *Power Spectral Density Functions*

$$\Gamma(\mathbf{r}) \Longleftrightarrow |\, \widetilde{\gamma}(\mathbf{k}) \,|^2$$

- ***Gaussian*** Random Medium

$$|\, \widetilde{\gamma}(\mathbf{k}) \,|^2 = \widetilde{\gamma}_0^2 \exp\left(-\frac{k^2}{k_0^2}\right)$$

- Random ***Fractal*** Medium

$$|\, \widetilde{\gamma}(\mathbf{k}) \,|^2 \sim \frac{1}{k^{2q}} \qquad \Gamma(\mathbf{r}) \sim \frac{1}{r^{3-q}}$$

Problem with the Weak Field Solution

- Solution depends on the condition

$$\frac{\|u_s\|}{\|u_i\|} << 1$$

which translates to: Wavelength >> V

- Incompatible with imaging systems in which the resolution is based on

Wavelength ~ V

Strong Scattering Model

$$\begin{gathered}\text{wavefield generated by single scattering events}\\+\\\text{wavefield generated by double scattering events}\\+\\\text{wavefield generated by triple scattering events}\\+\\\vdots\end{gathered}$$

Physically sound but mathematically speaking, a mess waiting to happen!!

Diffusion Based Model

- Consider multiple scattering events to be analogous to random walks of 'light rays' propagating between scattering sites:

$$D\nabla^2 I(\mathbf{r},t) = \frac{\partial}{\partial t} I(\mathbf{r},t), \quad \mathbf{r} \in V.$$

- Image plane solution (for an infinite domain) is

$$I(x,y,t) = \frac{1}{4\pi Dt} \exp\left[-\left(\frac{(x^2+y^2)}{4Dt}\right)\right] \otimes_2 I_0(x,y)$$

$$I(x,y,0) = I_0(x,y)$$

Wave to Diffusion Equation *Transformation*

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u = 0$$

$$u(x,y,z,t) = \phi(x,y,z,t)\exp(i\omega t)$$

$$u^*(x,y,z,t) = \phi^*(x,y,z,t)\exp(-i\omega t)$$

Conditional Equation

$$\nabla^2 u = \exp(i\omega t)\nabla^2\phi$$

$$\frac{\partial^2}{\partial t^2}u = \exp(i\omega t)\left(\frac{\partial^2}{\partial t^2}\phi + 2i\omega\frac{\partial\phi}{\partial t} - \omega^2\phi\right)$$

$$\simeq \exp(i\omega t)\left(2i\omega\frac{\partial\phi}{\partial t} - \omega^2\phi\right) \qquad \left|\frac{\partial^2\phi}{\partial t^2}\right| << 2\omega\left|\frac{\partial\phi}{\partial t}\right|$$

$$(\nabla^2 + k^2)\phi = \frac{2ik}{c}\frac{\partial\phi}{\partial t} \qquad (\nabla^2 + k^2)\phi^* = -\frac{2ik}{c}\frac{\partial\phi^*}{\partial t}$$

Diffusion Equation for the ***Intensity***

$$\phi^*\nabla^2\phi - \phi\nabla^2\phi^* = \frac{2ik}{c}\left(\phi^*\frac{\partial\phi}{\partial t} + \phi\frac{\partial\phi^*}{\partial t}\right)$$

$$\nabla^2 I - 2\nabla\cdot(\phi\nabla\phi^*) = \frac{2ik}{c}\frac{\partial I}{\partial t}$$

$$I = \phi\phi^* = |\ \phi\ |^2$$

Conditional Result

$$k = k_0 - i\kappa \text{ (i.e. } \omega = \omega_0 - i\kappa c)$$

$$D\nabla^2 I + 2\mathrm{Re}[\nabla \cdot (\phi \nabla \phi^*)] = \frac{\partial I}{\partial t}$$

$$\mathrm{Im}[\nabla \cdot (\phi \nabla \phi^*)] = -\frac{k_0}{c}\frac{\partial I}{\partial t}$$

$$D = c/2\kappa$$

$$\mathrm{Re}[\nabla \cdot (\phi \nabla \phi^*)] = 0$$

Justification

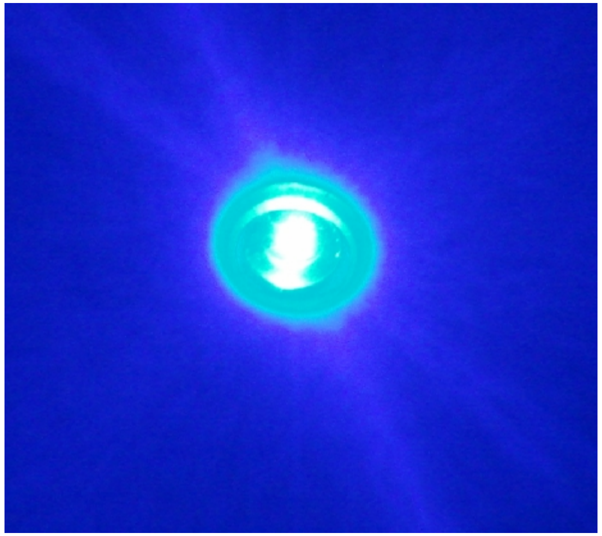
$$\mathrm{Re}\int_V \nabla\cdot(\phi\nabla\phi^*)d^3\mathbf{r} = \mathrm{Re}\oint_S \phi\nabla\phi^*\cdot\hat{\mathbf{n}}d^2\mathbf{r}$$

- Formally consider the surface to be at infinity so that diffusion occurs in the ***infinite domain***

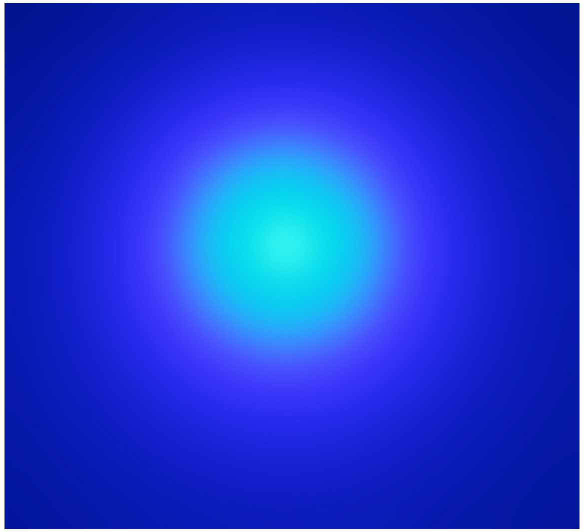
- Physically, light diffusers do not have a ***defined boundary***

Example:
Diffusion of Light Through Steam

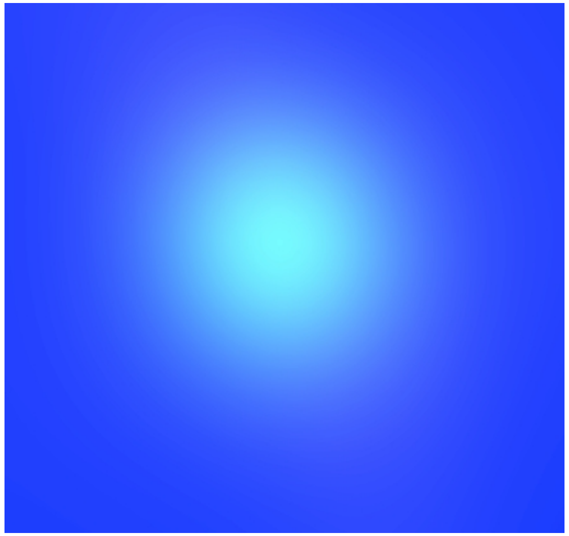
Source imaged through air



Source imaged through steam



Source convolved with a Gaussian PSF



Inverse Solution 1: Restoration of a Diffused Image

$$I(x,y,0) = I(x,y,T) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} T^n \left[\frac{\partial^n}{\partial t^n} I(x,y,t) \right]_{t=T}$$

$$\frac{\partial^2 I}{\partial t^2} = D\nabla^2 \frac{\partial I}{\partial t} = D^2 \nabla^4 I$$

$$\frac{\partial^3 I}{\partial t^3} = D\nabla^2 \frac{\partial^2 I}{\partial t^2} = D^3 \nabla^6 I$$

$$\left[\frac{\partial^n}{\partial t^n} I(x,y,t) \right]_{t=T} = D^n \nabla^{2n} I(x,y,T).$$

Inverse Solution 2: Restoration of a Diffused Image

$$I_0(x,y) = I(x,y,T) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}(DT)^n \nabla^{2n} I(x,y,T)$$

$$\sim I(x,y,T) - DT\nabla^2 I(x,y,T), \quad DT << 1.$$

High-Emphasis Filter

$$I_0(x,y) = I(x,y) - \nabla^2 I(x,y) \qquad DT = 1$$

$$\nabla^2 I_{ij} = I_{(i+1)j} + I_{(i-1)j} + I_{i(j+1)} + I_{i(j-1)} - 4I_{ij}$$

$$I_{ij}^0 = I_{ij} - \nabla^2 I_{ij} = 5I_{ij} - I_{(i+1)j} - I_{(i-1)j} - I_{i(j+1)} - I_{i(j-1)}$$

$$I_{ij}^0 \equiv I_0(i,j)$$

Finite Impulse Response (FIR) Filter: ***First Order***

$$I - \nabla^2 I$$

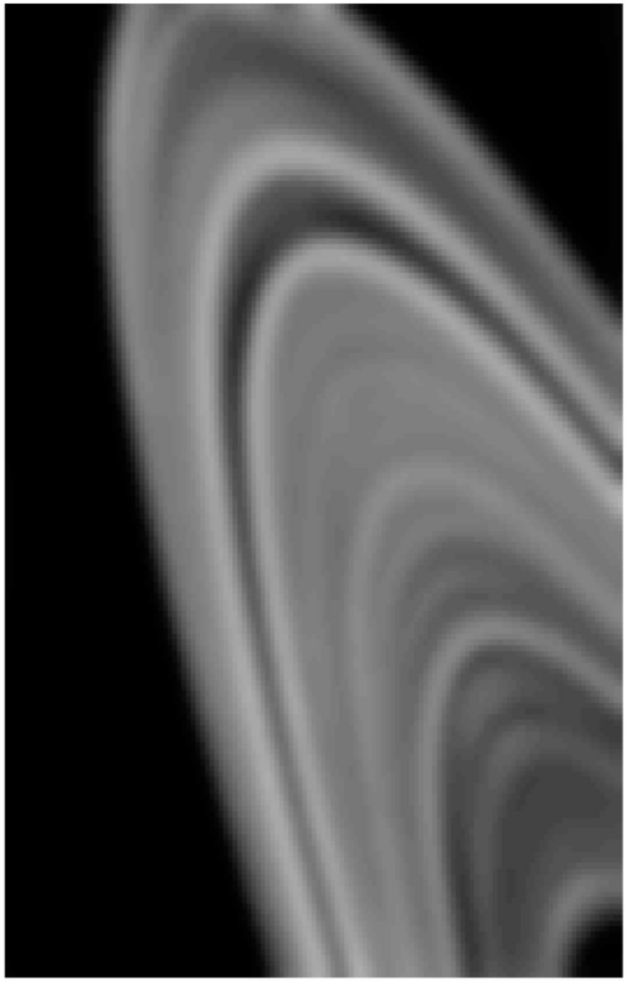
$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Finite Impulse Response (FIR) Filter: ***Second Order***

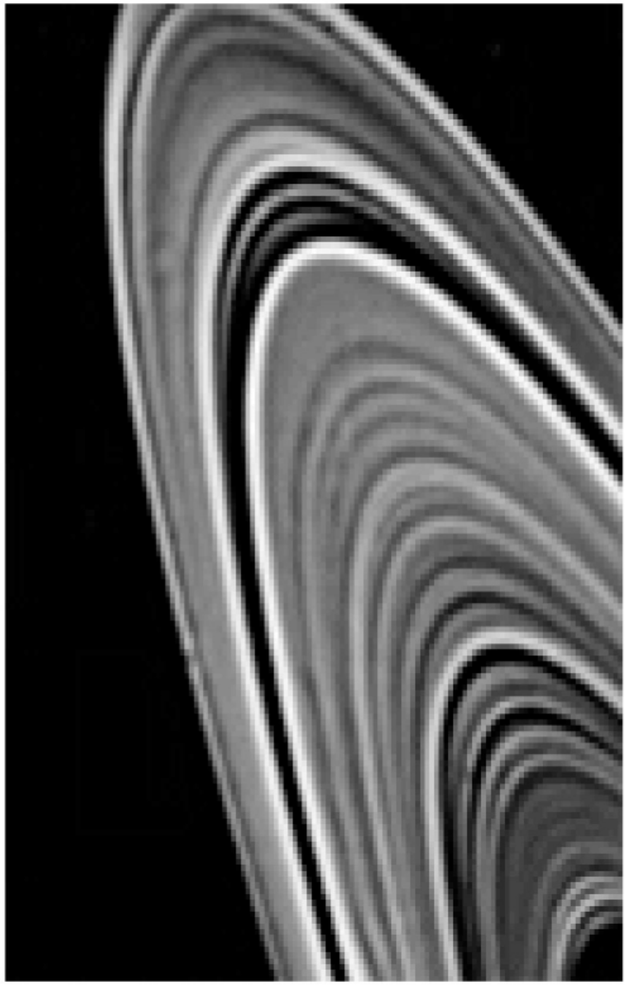
$$I - \nabla^2 I + \frac{1}{2}\nabla^4 I$$

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -10 & 2 & 0 \\ 1 & -10 & 30 & -10 & 1 \\ 0 & 2 & -10 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Example Application



$$I$$



$$I - \nabla^2 I$$

Case Study: *Fractional Diffusion Imaging*

- Problem: How can we model intermediate scattering processes (not weak or strong)

- Solution: Consider a fractional diffusion model compounded in the fractional PDE

$$\nabla^2 I(\mathbf{r},t) - \sigma^q \frac{\partial^q}{\partial t^q} I(\mathbf{r},t) = I_0(\mathbf{r},t)$$

$$D^q = 1/\sigma^q \qquad\qquad q \in [1,2]$$

Principal Aim

- Derive an **Optical Transfer Function** that models the effect of light scattering from a tenuous random medium

- Tenuous medium?

  ~ 10^6 light scattering particles m^{-3}

- **Goal of presentation:** To generate interest in the use of *fractional dynamics* for image synthesis, processing and analysis

'Stardust in Perseus'

http://apod.nasa.gov/apod/ap071129.html

Applications include processing Hubble Space Telescope images when light has propagated through cosmic dust

2D Green's Function Solution

$$I(\mathbf{r},t) = G(r,t) \otimes_2 \otimes_t I_0(\mathbf{r},t)$$

$$G(r,t) = \frac{1}{\sqrt{r}} \frac{1}{\sigma^{q/4} t^{1-q/4}} - \sqrt{r}\sigma^{q/4}\delta^{q/4}(t)$$

$$+\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} r^{(2n+1)/2} \sigma^{3nq/4} \delta^{3nq/4}(t)$$

Fractional diffusion equation and Green function approach: Exact solutions
E.K. Lenz et al, Physica A 360 (2006) 215–226

Asymptotic Solution

- For $\sigma \rightarrow 0,$

$$G(r,t) = \frac{1}{\sqrt{r}\sigma^{q/4}t^{1-q/4}}$$

- Spatial solution at any time *t* is given by

$$I(x,y) = \frac{1}{(x^2+y^2)^{\frac{1}{4}}} \otimes_2 I_0(x,y)$$

A Fractal 'Impulse Response Function'

$$G(r,t) = \frac{1}{\sqrt{r}\sigma^{q/4}t^{1-q/4}}$$

If the source function is white noise, then:

- Temporal component of the Green's function yields random fractal noise
- Spatial component of Green's function yields a random fractal surface (Mandelbrot surface) with a Fractal Dimension of 2.5

Deconvolution

$$I(x,y) = p(x,y) \otimes_2 I_0(x,y) + n(x,y)$$

- Problem: Find an estimate for $I_0(x,y)$
- Assumption: $\Pr[n(x,y)]$ is Gaussian

Strong scattering in a random medium

$$p(x,y) = \exp\left[-\left(\frac{x^2+y^2}{4DT}\right)\right]$$

Strong scattering in a tenuous random medium

$$p(x,y) = \frac{1}{(x^2+y^2)^{\frac{1}{4}}}$$

A Bayesian Inverse Filter

- A Gaussian noise assumption yields the following *a Posteriori* (inverse) filter

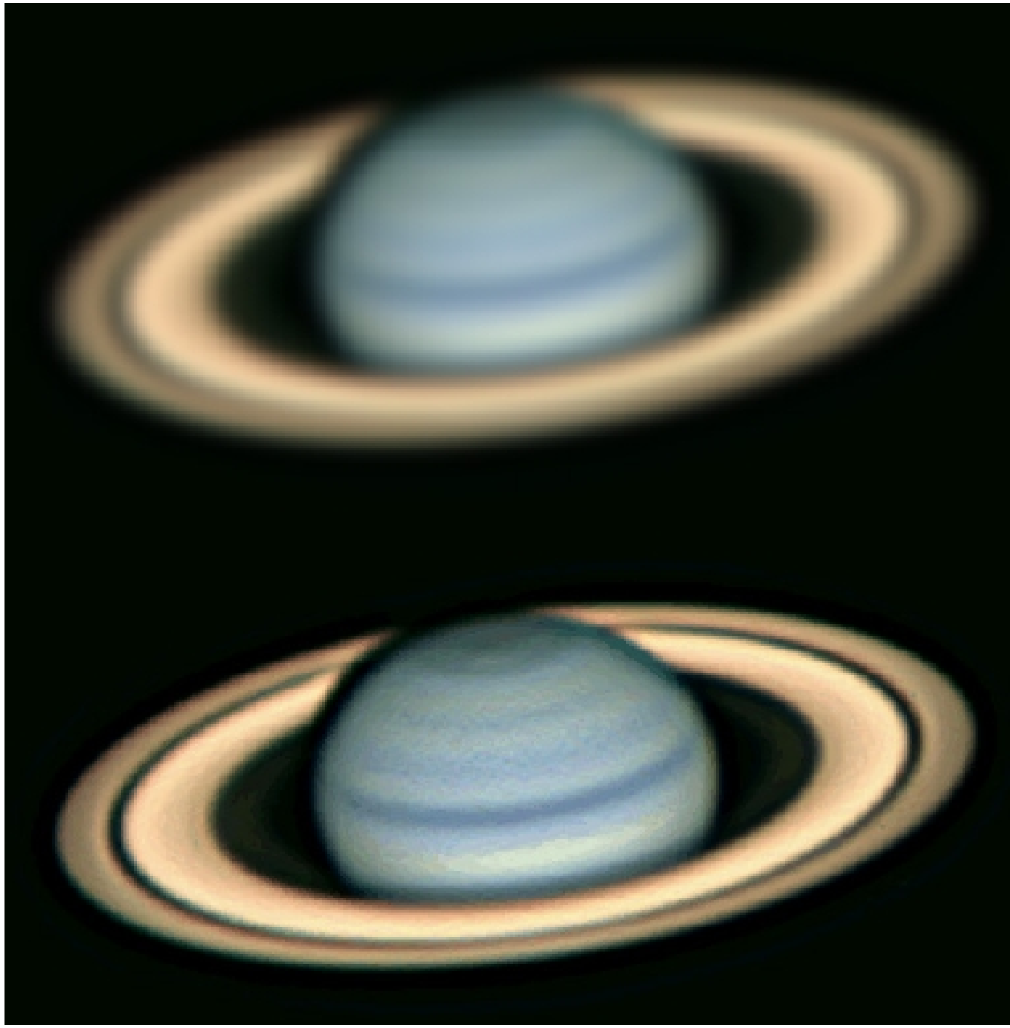
$$F(k_x, k_y) = \frac{P^*(k_x, k_y)}{|\ P(k_x, k_y)\ |^2 + \sigma_n^2 / \sigma_{I_0}^2}$$

where σ_{I_0} / σ_n defines the SNR

- Adaptive filtering is used based on searching for a reconstruction with a ***maximum average gradient*** for ***minimum zero crossings***

Example Results

Diffusion



Fractional Diffusion

Light Management

Application of fractional diffusion imaging in

Light Management Technology:

Quality control for mass production of diffusers

http://www.microsharp.co.uk

Microsharp
Corporation Limited

Manufacture of Microsharp Light Diffusers based on q

Microsharp Corporation Limited

- Light diffusion based model

$$I_0(x,y) = I(x,y,T) - DT\nabla^2 I(x,y,T)$$

- Fractional light diffusion based model

$$I_0(x,y) = I(x,y,T) - \frac{D^q T}{\Gamma(q)}\nabla^2 I(x,y,T)$$

Diffusion and Fractional Diffusion based Models for Multiple Light Scattering and Image Analysis, J M Blackledge, ISAST Trans. in Electronics and Signal Processing, ISSN 1797-2329, No. 1, Vol. 1, 38 - 60, 2007

Summary

- The Diffusion Equation has been used to model strong scattering processes

- The inverse scattering problem reduces to: *'Deconvolution for a Gaussian PSF'*

- We have modelled intermediate scattering using a ***Fractional Diffusion Equation*** and shown that, for a highly diffuse medium, the ***Optical Transfer Function*** is

$$(k_x^2 + k_y^2)^{-0.75}$$

Open Problems

- What is the effect of including further terms in the fractional Green's function?

  i.e. can we produce an OTF that:

  - does not rely on an asymptotic solution

  - is of practical value (e.g. for deconvolution)

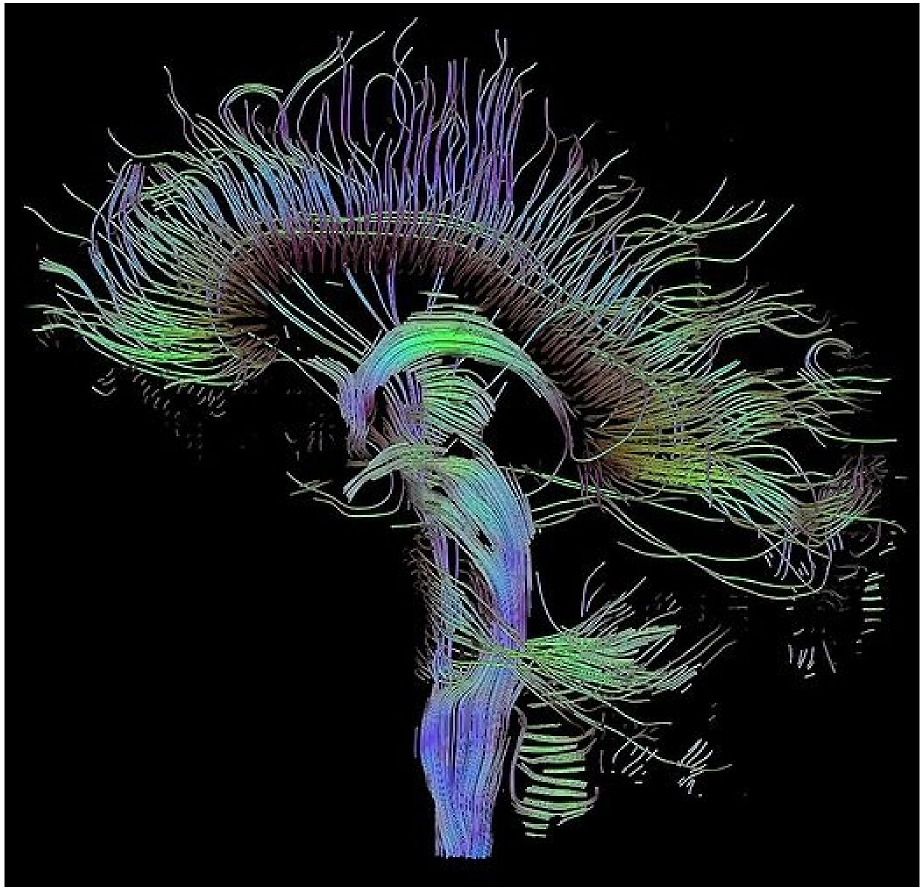
- Method applies to incoherent imaging only, i.e. a fractional diffusion equation for the **intensity.**

How can we use the same approach to model intermediate coherent scattering?

Diffusion MRI

$$\mathbf{D}\nabla^2 I - \frac{\partial I}{\partial t} = 0$$

$$\mathbf{D} = \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix}$$



Questions

Contents of Presentation

- Light Scattering from Random Media
  - Weak scattering model
  - Strong scattering model
  - Diffusion based model for multiple scattering
- Fractional Diffusion: A Model for Intermediate Scattering
- A Filter using Bayesian Estimation
- Some Example Results
- Questions