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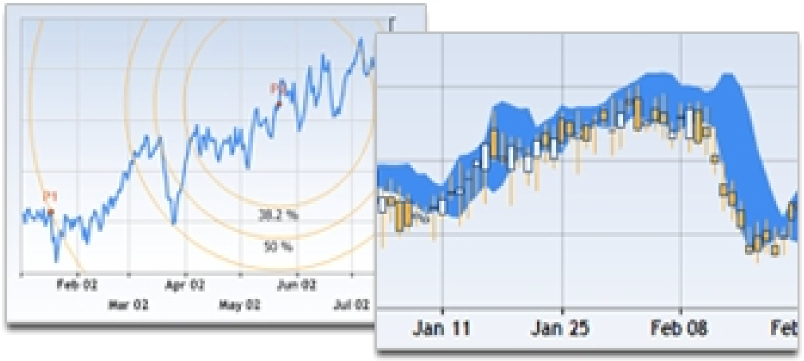

Friday 5th March, 2010: 11:00-13:00

BANK OF ENGLAND

Financial Modelling using the Fractal Market Hypothesis

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HUMAN CAPITAL
NATIONAL COHESION STRATEGY

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Lectures co-financed by the European Union in scope of the European Social Fund

What is the Problem?

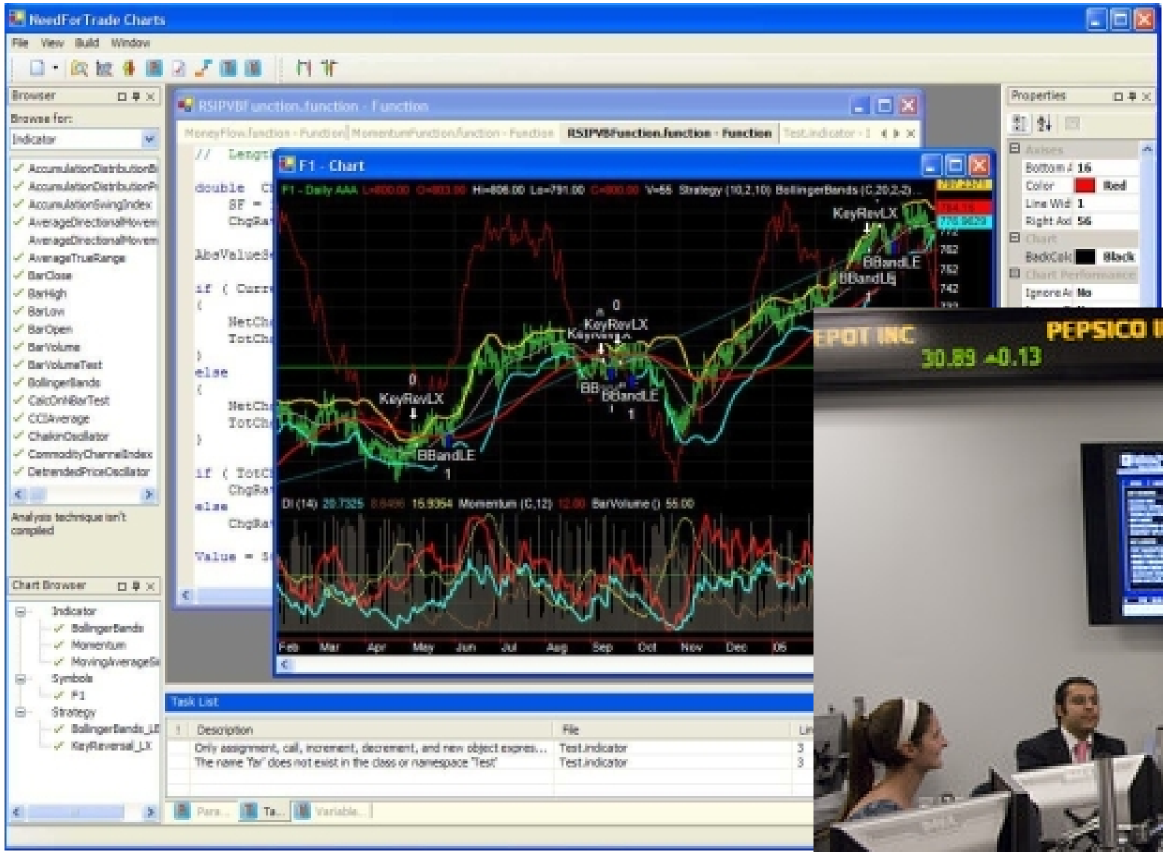
- Financial indices are digital signals composed of ***tick data*** corresponding to different measures of an economy over a range of scales

- Is it possible to analyse these signals in such a way that a prediction can be made on their likely future behaviour or *trend* ?

  - Time Series Forecasting

  - Systemic Risk Assessment

Financial Risk Management ?



What is Capitalism?

Greed **Fear**

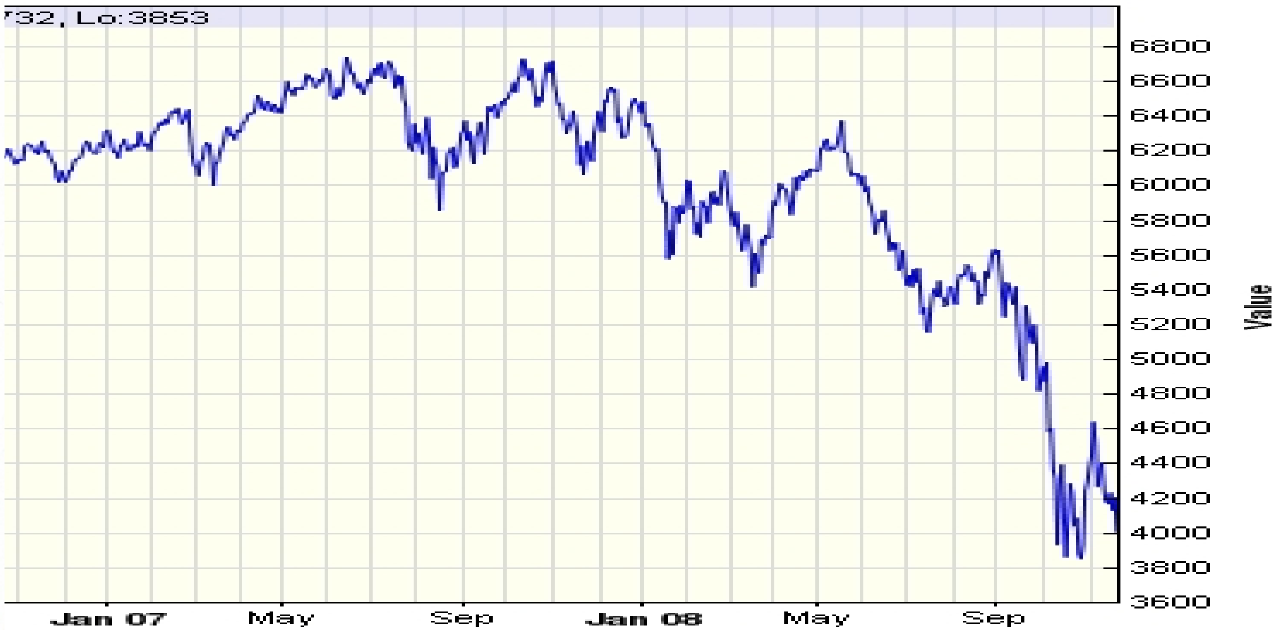
Balance

The Price of Greed

Greed

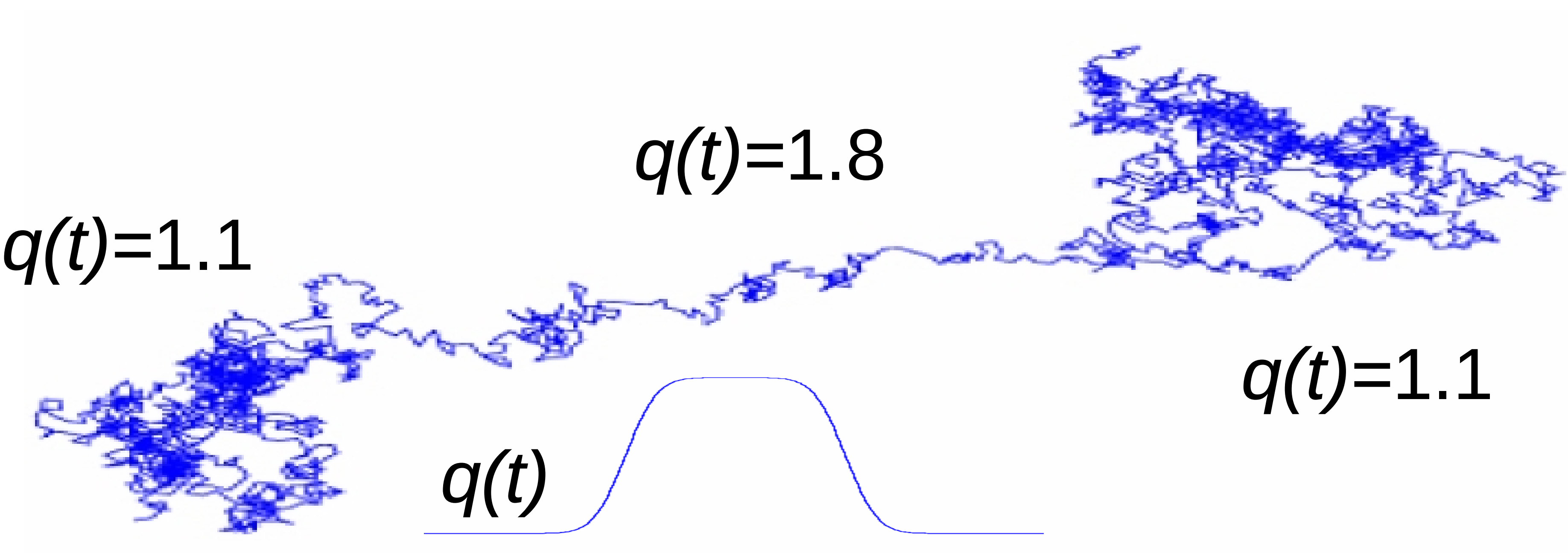
Fear

Balance



FMH Approach to a Solution

The FMH operator: $$\frac{\partial^2}{\partial x^2} - \sigma^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}}$$



Hypothesis: A change in *q(t)* <u>precedes</u> a change in a financial signal

Principal Papers

Application of the Fractal Market Hypothesis for Modelling Macroeconomic Time Series
Blackledge J M, ISAST Transactions on Electronics and Signal Processing, No.1, Vol. 2, 89-110, 2008
ISSN:1797-2329 **http://eleceng.dit.ie/papers/106.pdf**

Systemic Risk Assessment using a Non-stationary Fractional Dynamic Stochastic Model for the Analysis of Economic Times Series, Blackledge J M, ISAST Transactions on Electronics and Signal Processing, 2010 (Submitted) **http://eleceng.dit.ie/papers/148.pdf**

Contents of Presentation I

Part I:

- What are Fractals?
- Random Walks and Hurst Processes
- Random Walks and the Fractional Diffusion Equation
- Fractional Calculus
- Basic Model for a Stationary Process
- Levy statistics and the Fractional Diffusion Equation
- Questions
- Interval (10 Minutes)

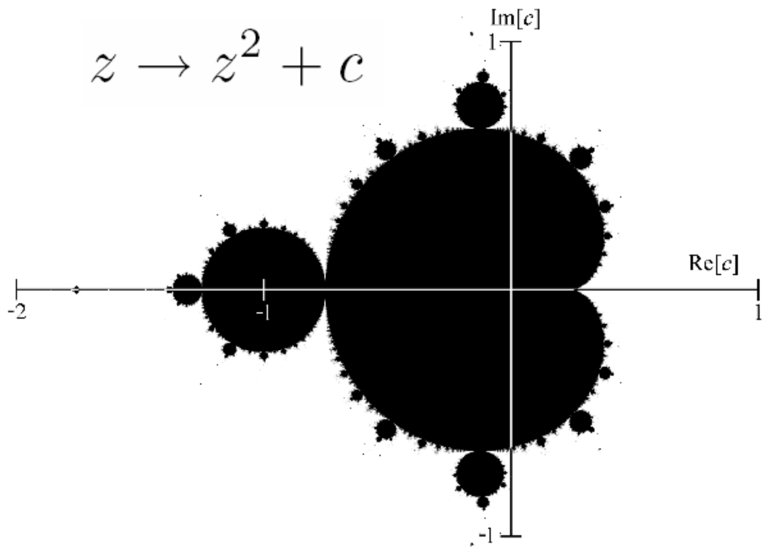
Contents of Presentation II

Part II:

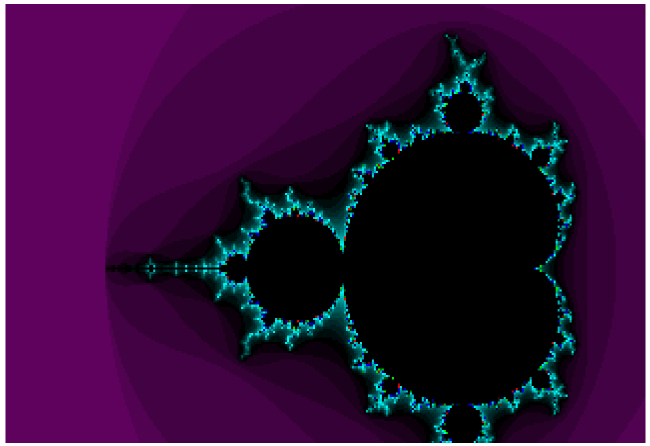
- The Efficient Market Hypothesis
- Properties of Financial Signals
- The Fractal Market Hypothesis
- Numerical Algorithms
- Example Results
- Case Study: ABX index analysis for the Bank of England
- Summary and Research Project Proposals
- Questions

Part 1: *What are Fractals ?*

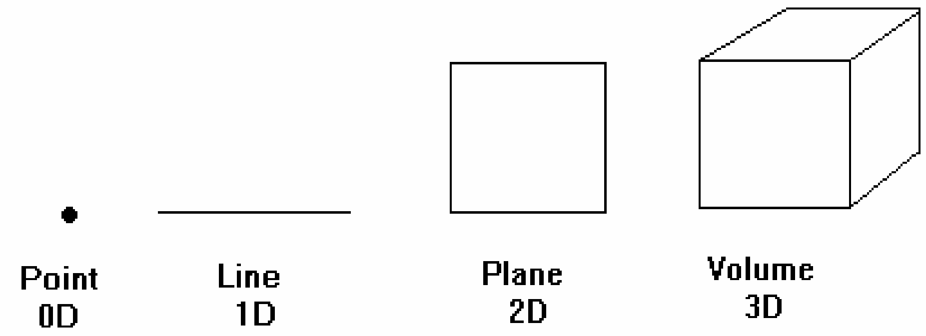
"The term ***fractal*** is derived from the Latin adjective *fractus.* The corresponding Latin verb *frangere* means 'to break', to create irregular fragments. In addition to 'fragmaneted' *fractus* should also mean 'irregular', both meanings being preserved in *fragment".*

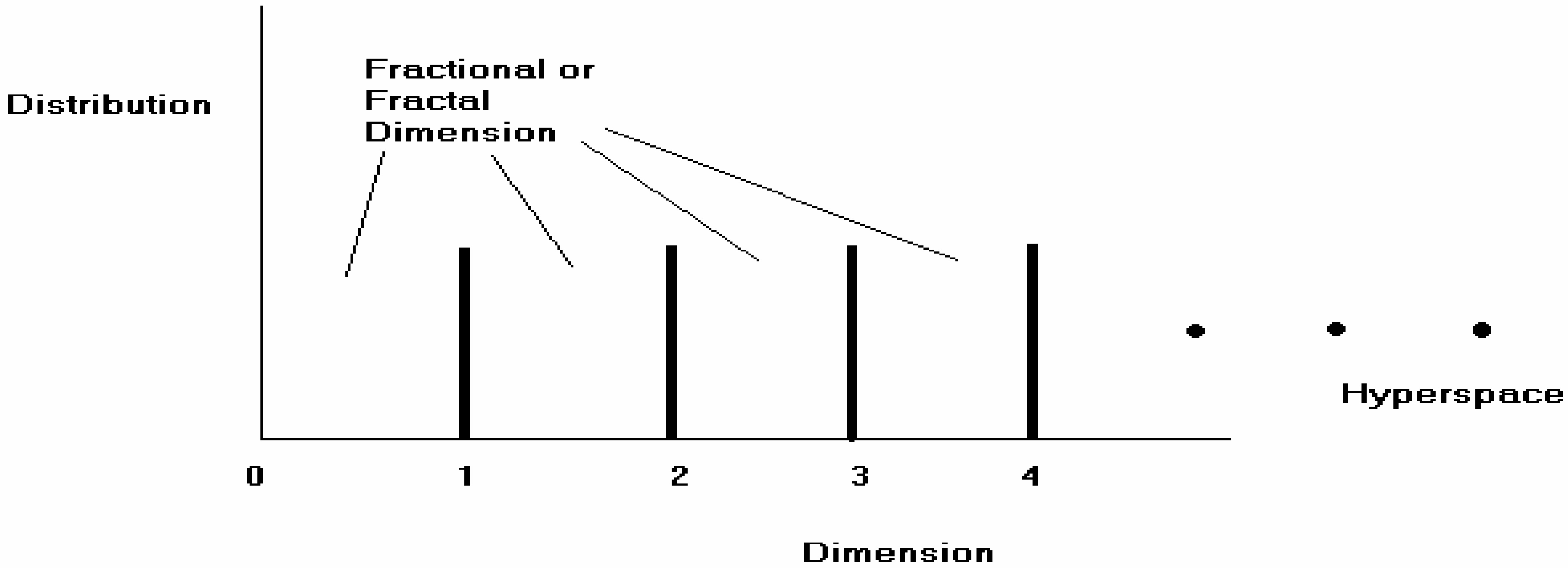


B Mandelbrot



Euclidean & Fractal Dimensions





Fundamental Definition of the Fractal Dimension

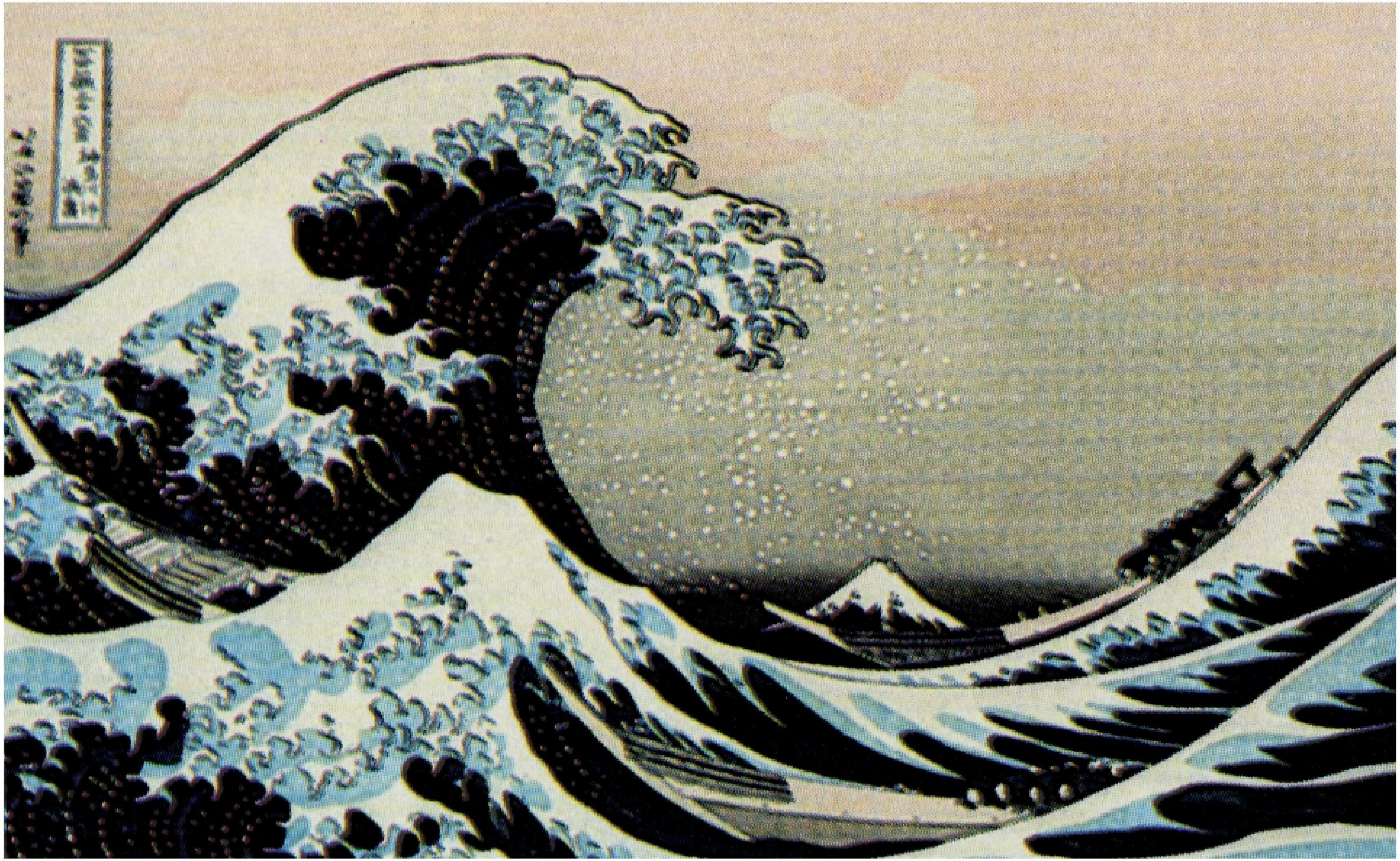
$$D = -\frac{\log(N)}{\log(r)}$$

Fractal Types

Fractal type	Fractal Dimension
Fractal Dust	$0 < D < 1$
Fractal Curve	$1 < D < 2$
Fractal Surface	$2 < D < 3$
Fractal Volume	$3 < D < 4$
Fractal Time	$4 < D < 5$
Hyper-fractals	$5 < D < 6$
$\vdots$	$\vdots$

Underlying Philosophy

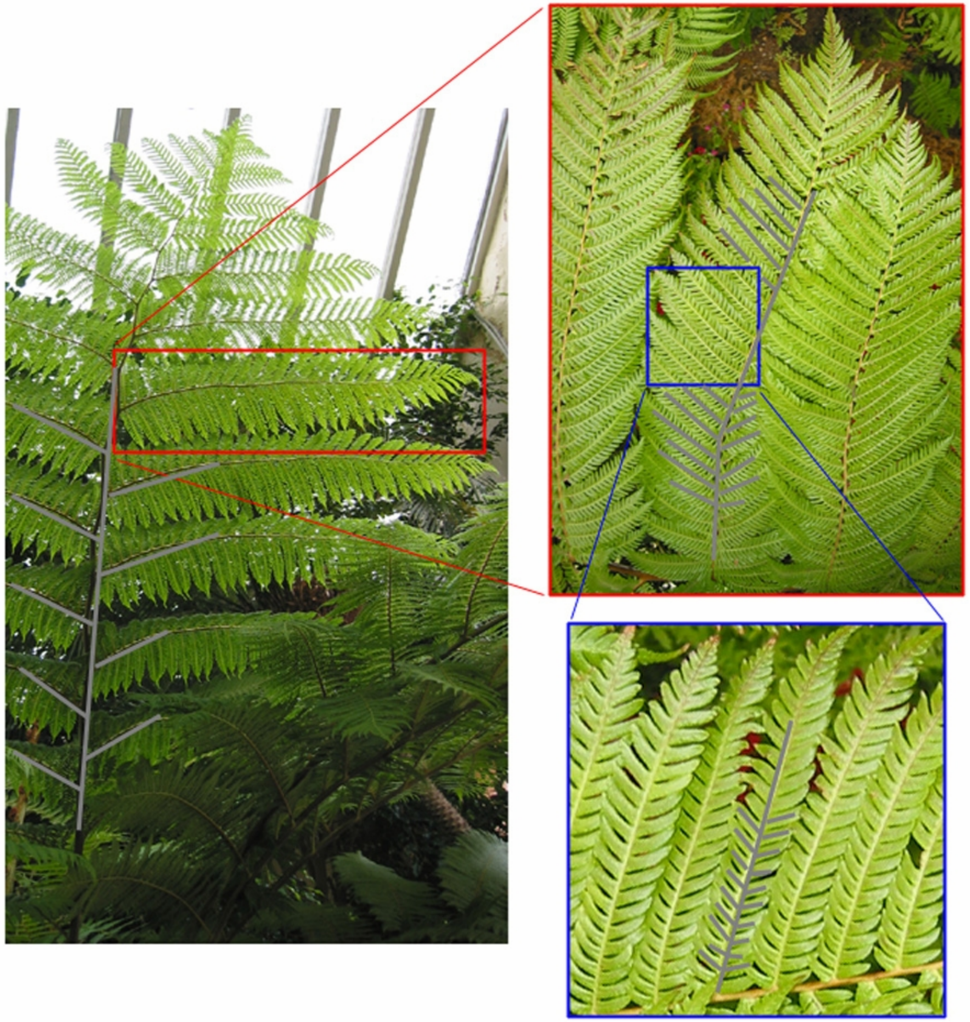
In every way one can see the shape of the sea



Self-Affine Structures

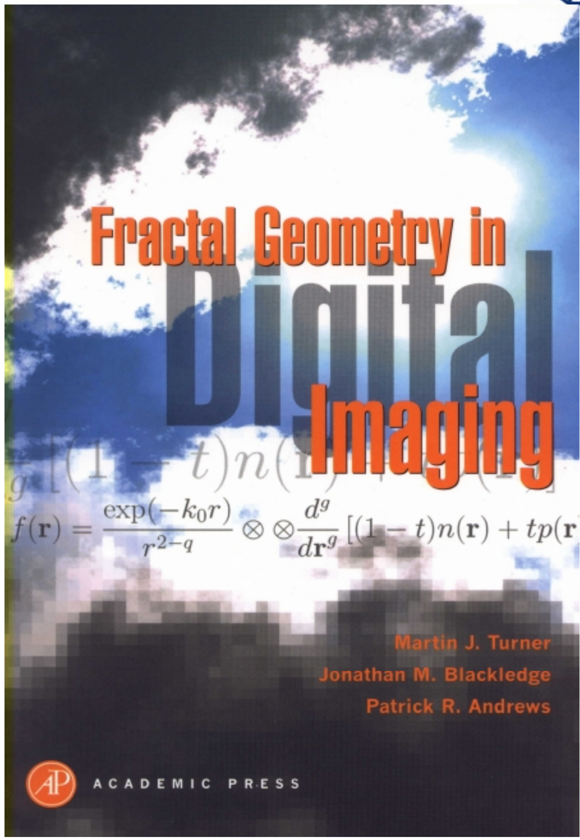
$$\lambda^q f(\mathbf{r}) = f(\lambda \mathbf{r})$$

$$q = 1 - D + \frac{3}{2} D_T$$



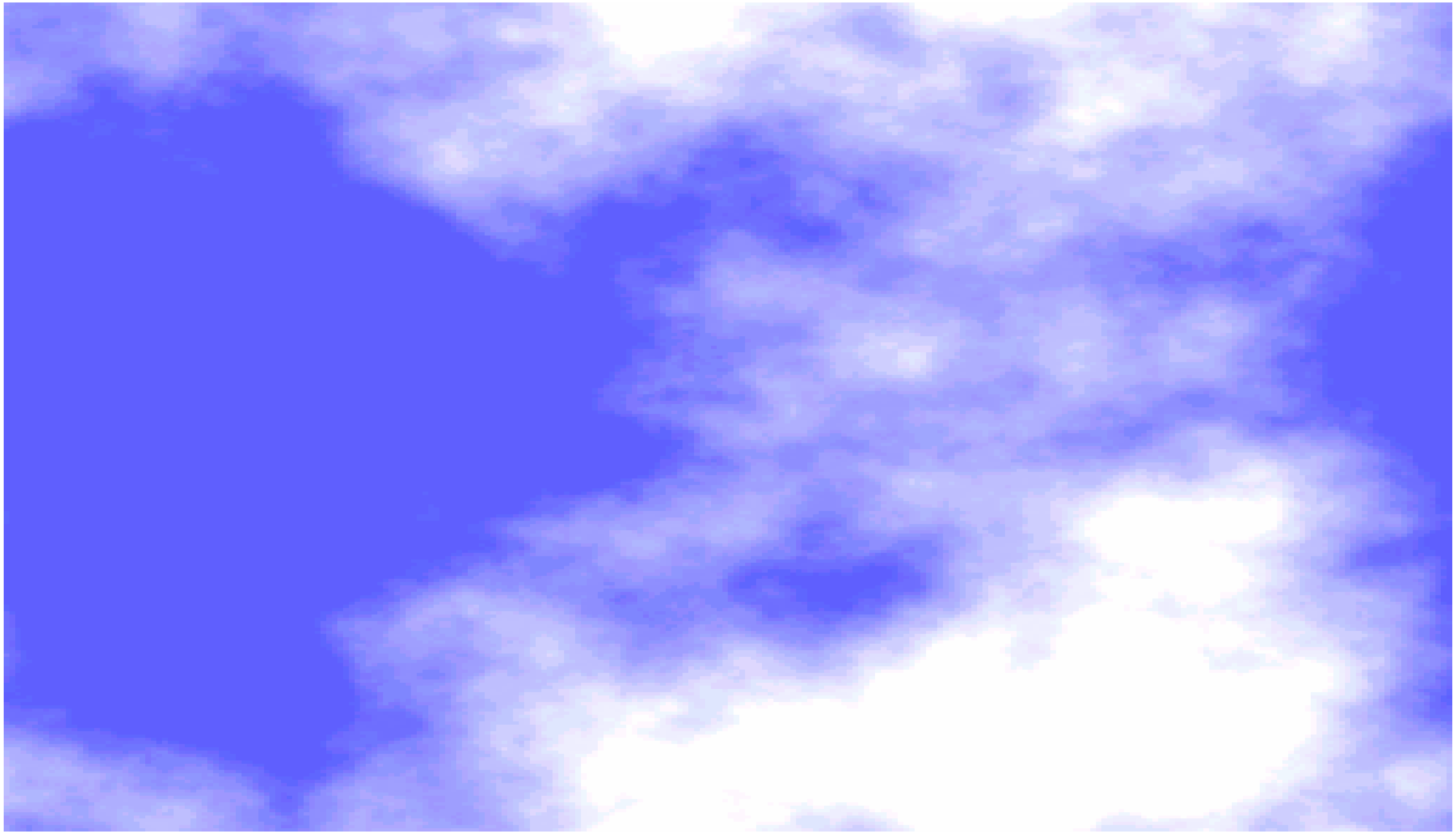
Fractals and Texture

$$\lambda^q \Pr[f(\mathbf{r})] = \Pr[f(\lambda \mathbf{r})] \qquad q = 1 - D + \frac{3}{2} D_T$$

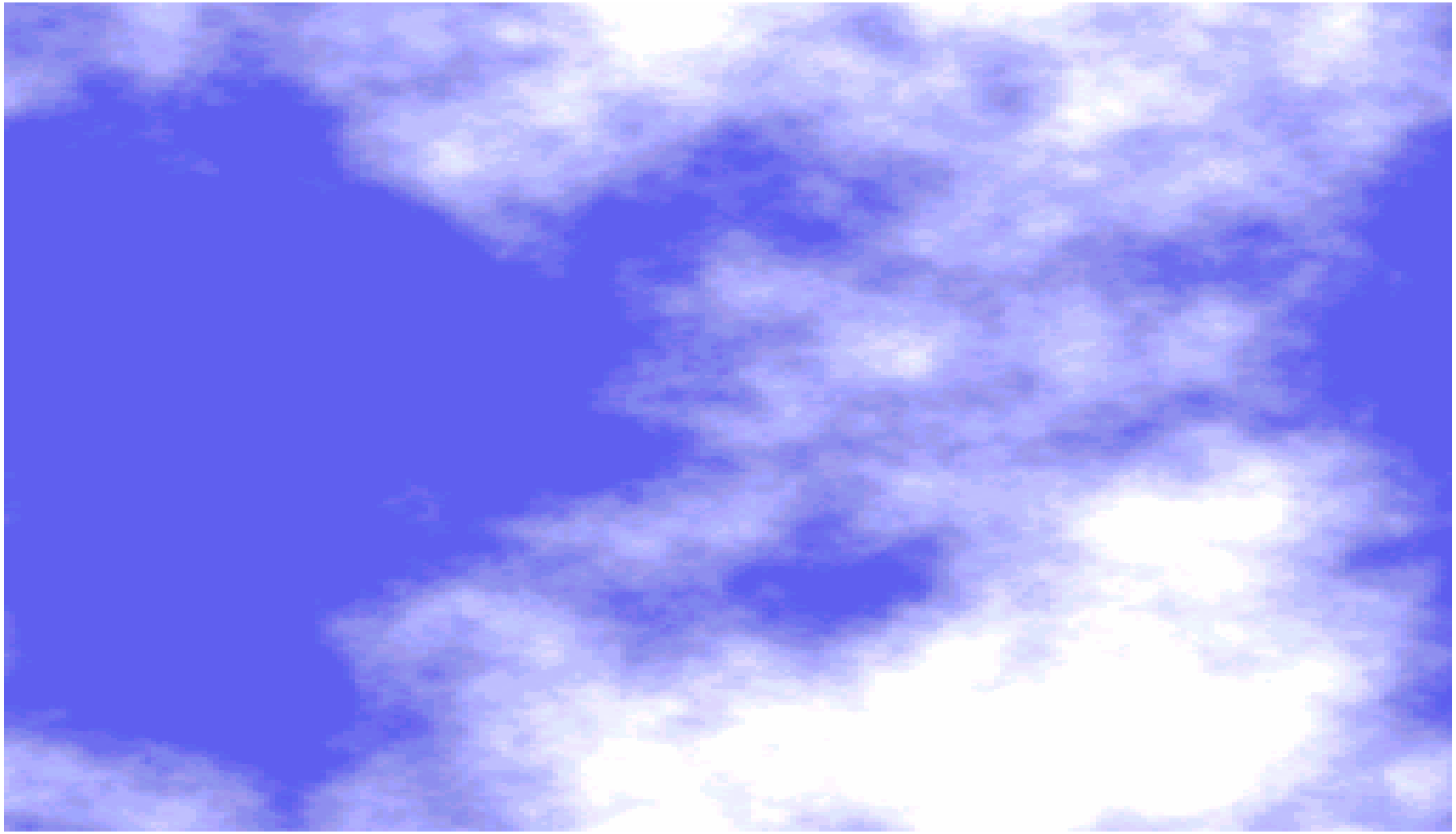


"Much of Fractal Geometry can be considered to be an intrinsic study of texture" *B Mandelbrot*

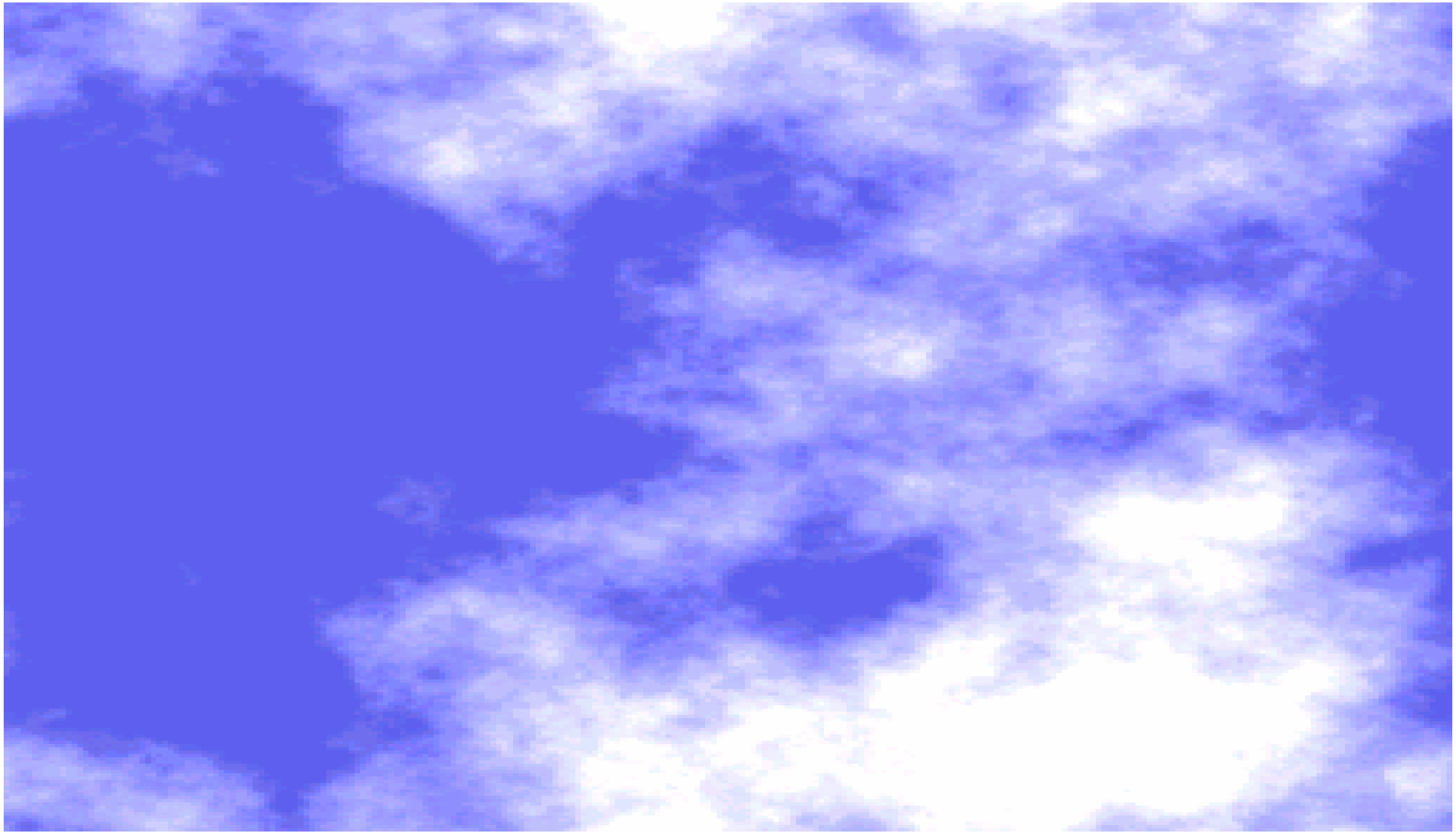
Fractal Clouds: D=2.1



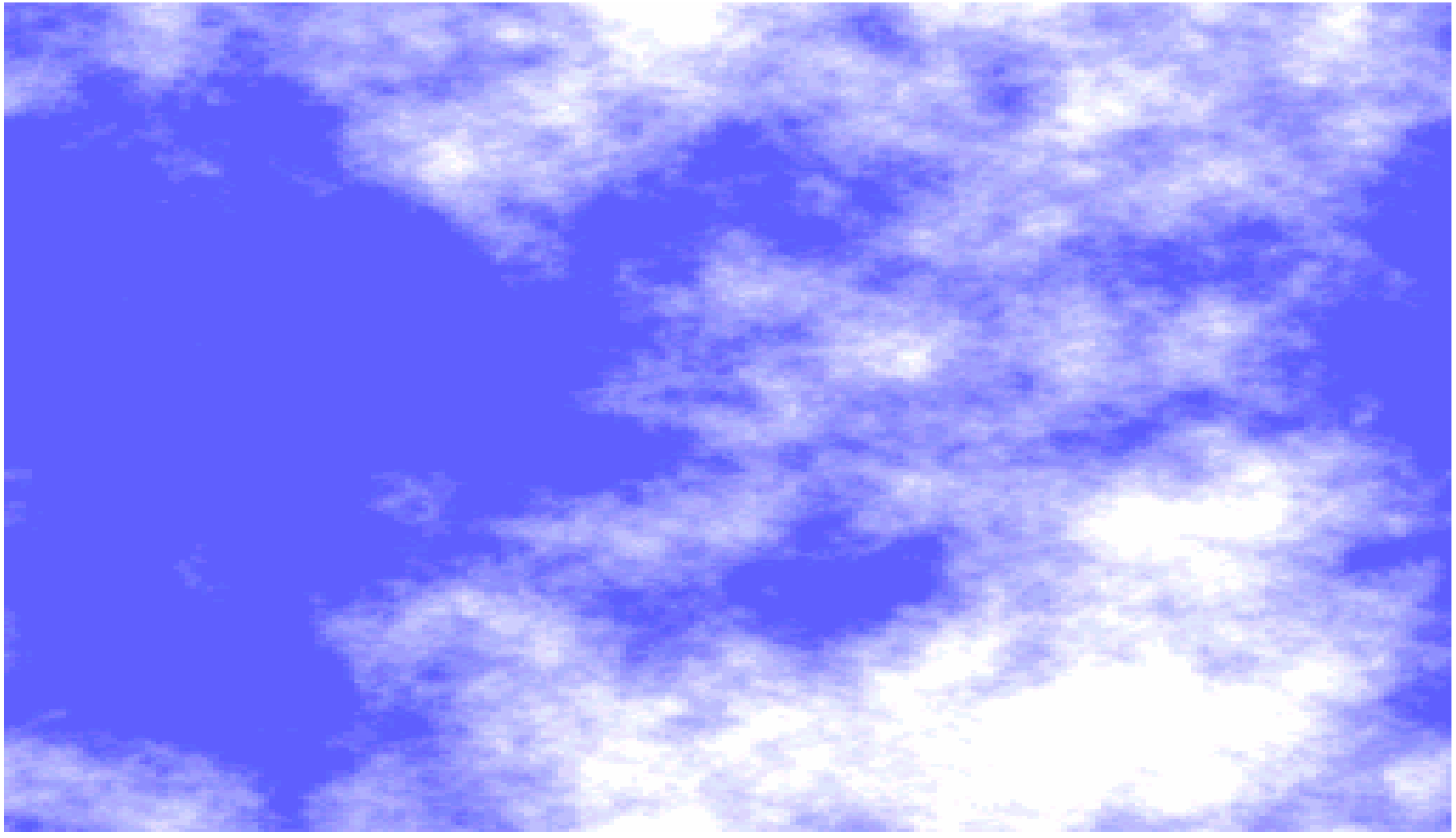
Fractal Clouds: D=2.2



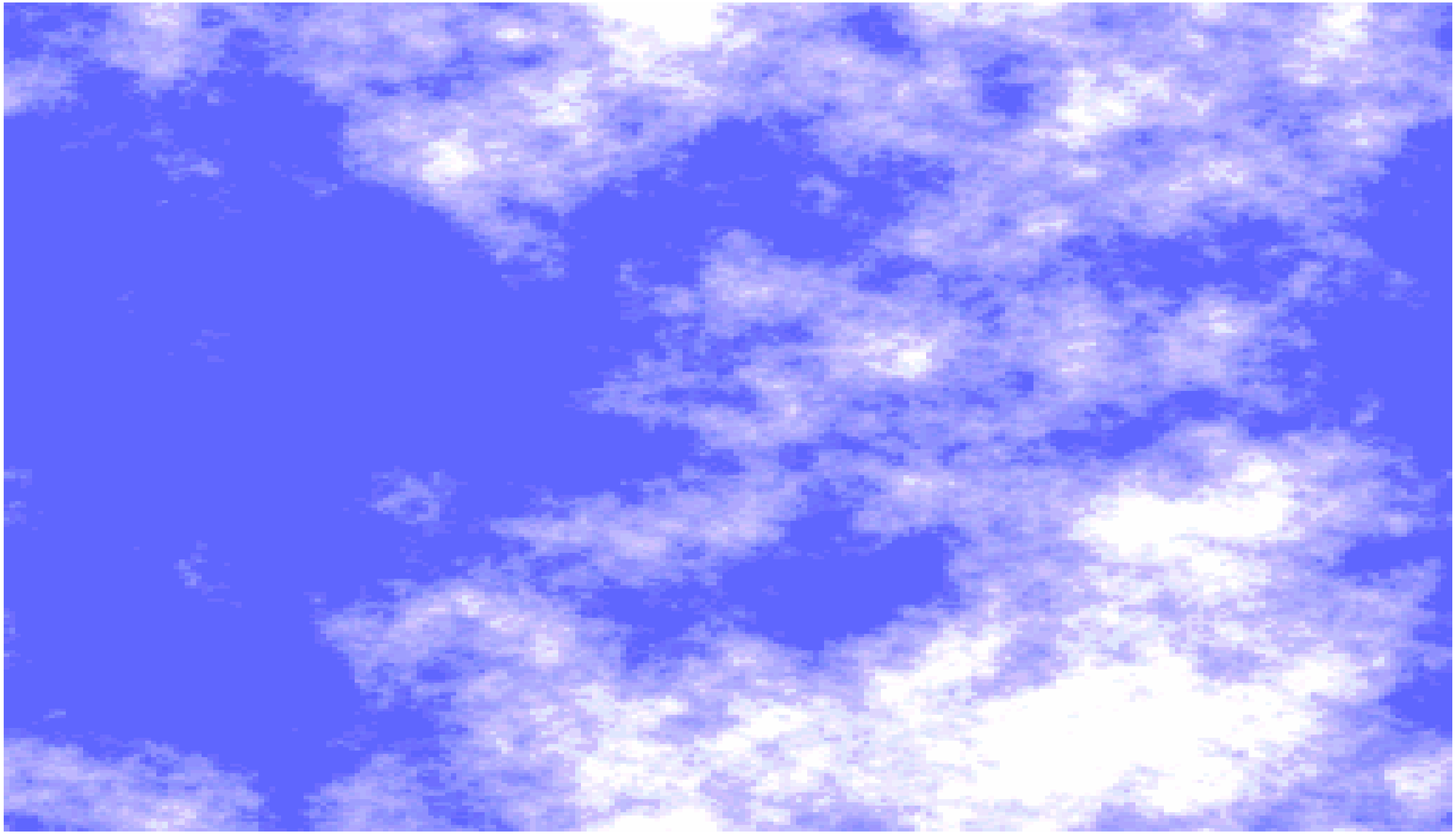
Fractal Clouds: D=2.3



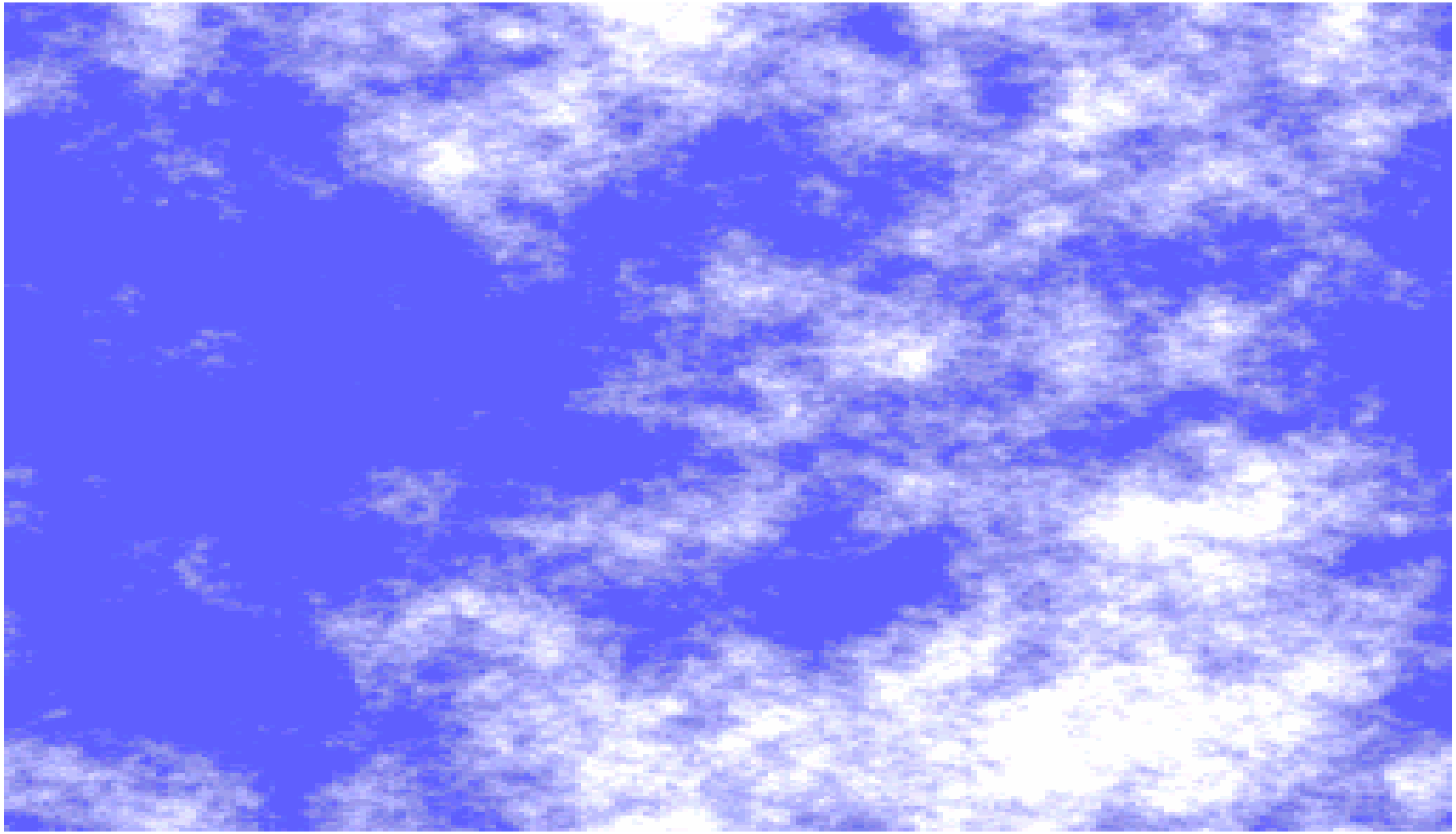
Fractal Clouds: D=2.4



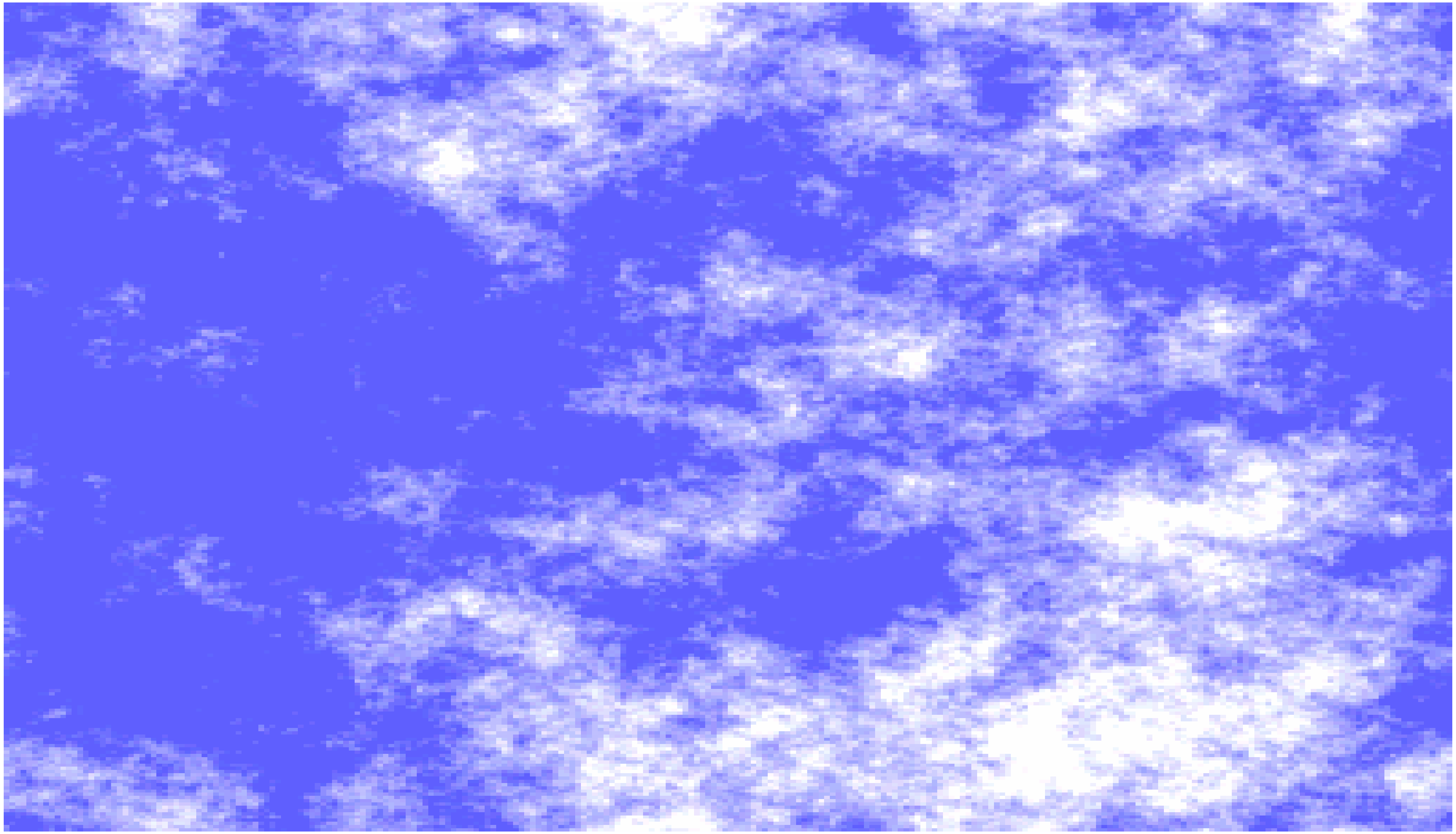
Fractal Clouds: D=2.5



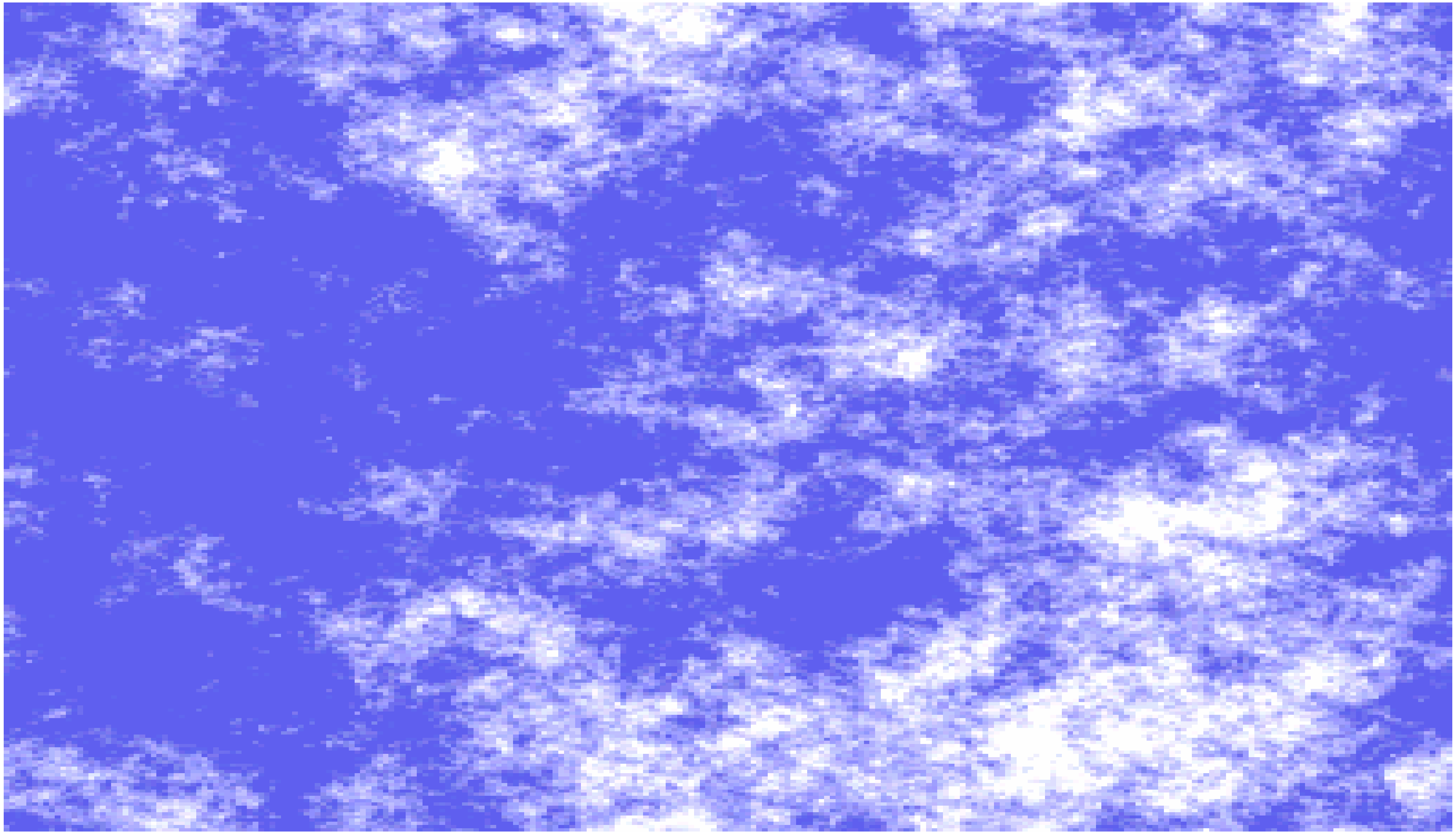
Fractal Clouds: D=2.6



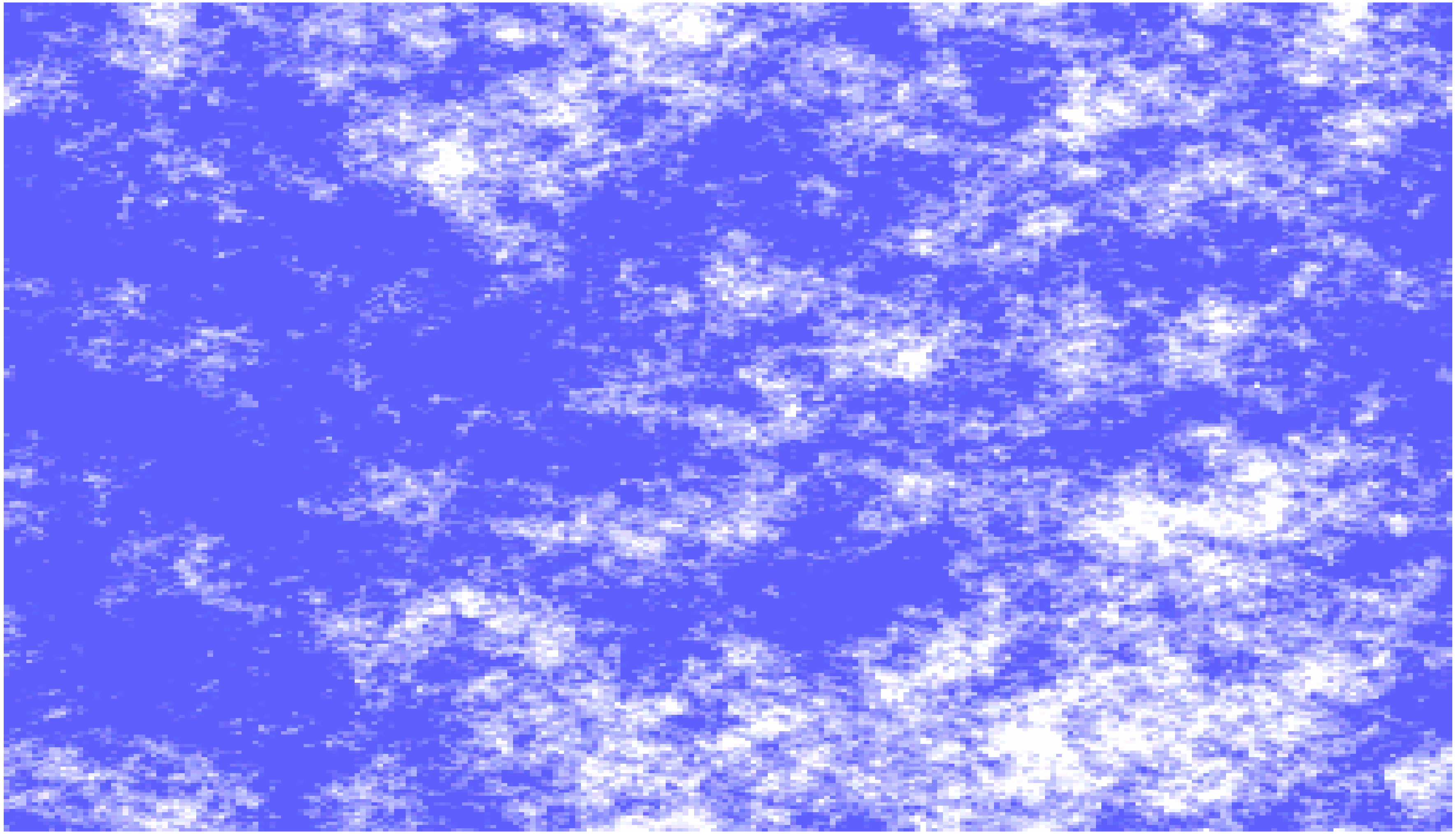
Fractal Clouds: D=2.7



Fractal Clouds: D=2.8



Fractal Clouds: D=2.9



Random Walks & Hurst Processes

$$H \sim 0.7$$

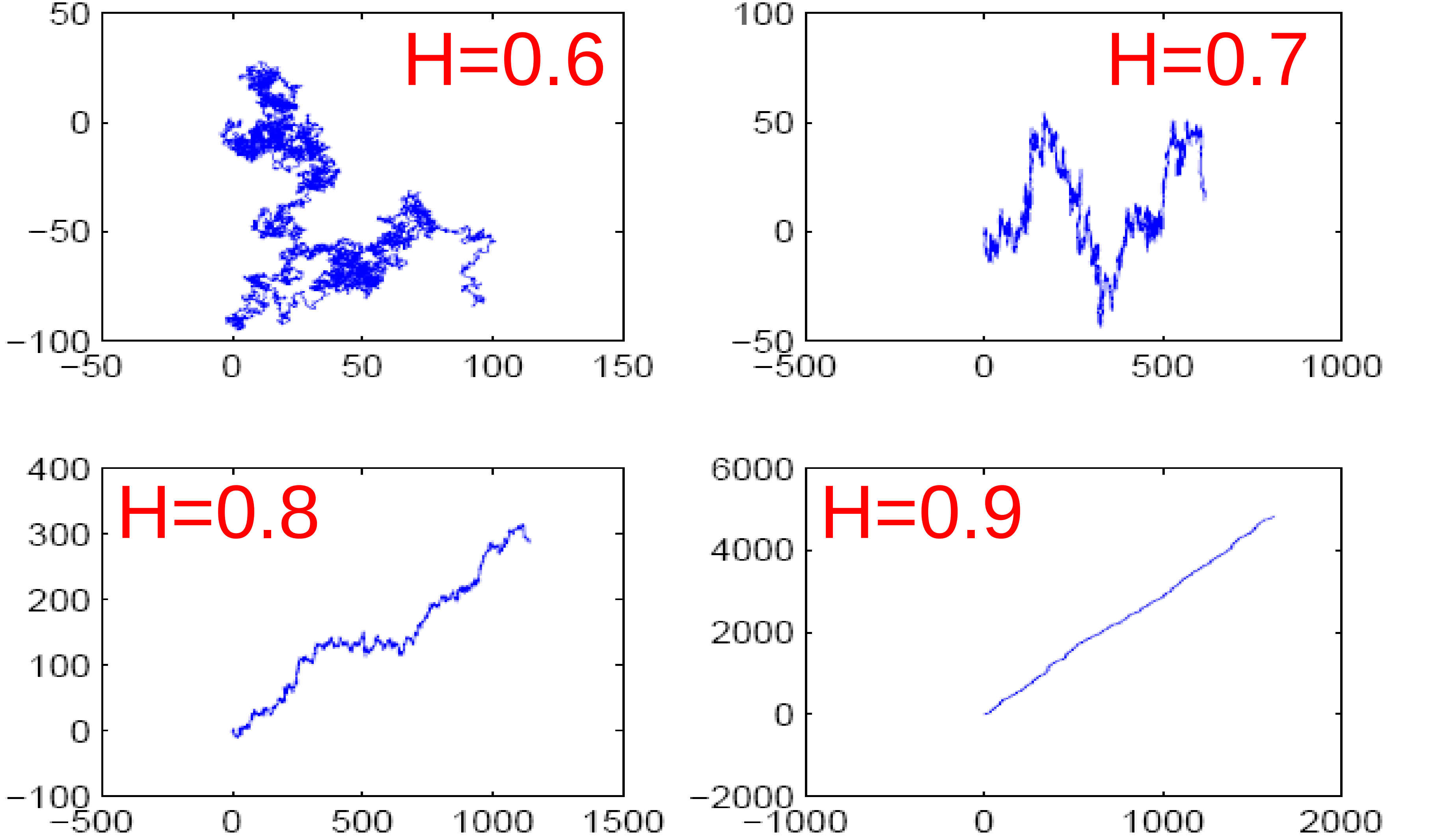
Brownian Motion

$$R(t) = a\sqrt{t}$$

$$R(t) = at^H$$

$$0 < H < 1$$

Random Fractal Walks with a Variable Hurst Exponent



Random Walks and the Fractional Diffusion Equation

$$R(t) = at^H \qquad \Longrightarrow \frac{\partial^2}{\partial x^2} - \sigma^q \frac{\partial^q}{\partial t^q}$$

Hurst exponent H:
0<H<1, D=2-H

Fourier dimension:
0<q<2; D=(5-2q)/2 (0.5<q<1.5)

- q=1: Diffusion Equation
- q=2: Wave Equation
- 1<q<2: Fractional (Fractal) Diffusion Equation

H>0.5 (q>1): random walk with **persistence and directional bias**

Fractional Calculus

- L'hospital to Leibnitz (1695):

  'Given that $d^n f/dt^n$ exists for all integer n, what if n be 1/2'.

- Leibnitz to L'hospital:

  'It will lead to a paradox ... From this paradox, one day useful consequences will be drawn'.

Fractional Integration

$$\frac{d^{-q}}{dt^{-q}} f(t) \Longleftrightarrow \frac{F(\omega)}{(i\omega)^q}$$

$$\frac{d^{-q}}{dt^{-q}} f(t) = \mathcal{F}^{-1}\left[\frac{F(\omega)}{(i\omega)^q}\right] = \frac{1}{\Gamma(q)t^{1-q}} \otimes f(t)$$

Fundamental Property: *Statistical self-affinity*

$$f'(t) = \frac{1}{\Gamma(q)t^{1-q}} \otimes f(\lambda t) = \frac{1}{\Gamma(q)} \int \frac{f(\lambda\tau)d\tau}{(t-\tau)^{1-q}}$$

$$= \frac{1}{\lambda^q} \frac{1}{\Gamma(q)} \int \frac{f(x)dx}{(\lambda t - x)^{1-q}} = \frac{1}{\lambda^q} f(\lambda t) \quad (x = \lambda\tau)$$

$$\Longrightarrow \lambda^q \Pr[f(t)] = \Pr[f(\lambda t)]$$

Fractional Integral Transforms

- Riemann-Liouville transform

$$\hat{I}^q f(t) = \frac{1}{\Gamma(q)} \int_{-\infty}^{t} \frac{f(\tau)}{(t-\tau)^{1-q}} d\tau, \quad q > 0$$

- Erdelyi-Kober transform
  (a generalised fractional integral)

$$\hat{I}^q f(t) = \frac{t^{-p-q+1}}{\Gamma(q)} \int_{0}^{t} \frac{\tau^{p-1}}{(t-\tau)^{1-q}} f(\tau) d\tau, \quad q > 0, \quad p > 0$$

Basic Model for a Stationary Process

Let *n(t)* be a ***'white noise'*** source and

$$\left(\frac{\partial^2}{\partial x^2} - \sigma^q \frac{\partial^q}{\partial t^q}\right) u(x,t) = \delta(x) n(t)$$

$$\frac{\partial^q}{\partial t^q} u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(x,\omega)(i\omega)^q \exp(i\omega t) d\omega$$

Transformation Equation

$$\left(\frac{\partial^2}{\partial x^2} + \Omega_q^2\right) U(x,\omega) = \delta(x) N(\omega)$$

$$\Omega_q = i(i\omega\sigma)^{\frac{q}{2}}$$

$$n(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\omega) \exp(i\omega t) d\omega$$

Green's Function Solution

$$U(x,\omega) = N(\omega)g(|\ x\ |,\omega)$$

$$g(|\ x\ |,\omega) = \frac{i}{2\Omega_q}\exp(i\Omega_q\ |\ x\ |)$$

Power Spectrum Characteristics

$$| U(x,\omega) |^2 = \frac{| N(\omega) |^2}{4\sigma^q \omega^q}, \quad \omega > 0$$

log(*Power Spectral Density Function*)

= log(*constant*) + *q* log(*frequency*)

Fundamental Solution (ignoring scaling constants)

$$u(x,t) = n(t) \otimes \frac{1}{t^{1-q/2}} + i \mid x \mid n(t)$$

$$+ \sum_{k=1}^{\infty} \frac{i^{k+1}}{(k+1)!} \mid x \mid^{2k} \frac{d^{kq/2}}{dt^{kq/2}} n(t)$$

$$u(t) = \frac{1}{t^{1-q/2}} \otimes n(t)$$

$$x \to 0$$

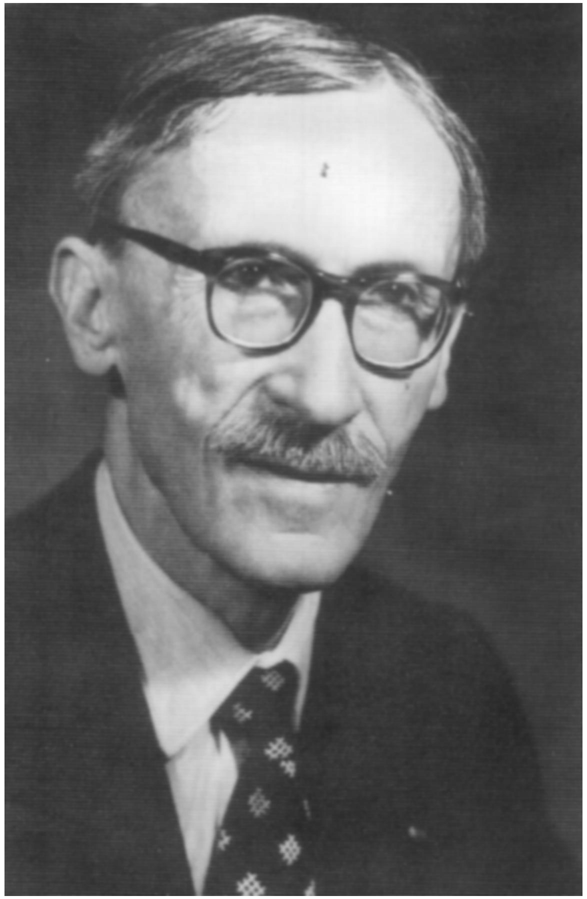
$$\lambda^{q/2} \Pr[u(\lambda t)] = \Pr[u(t)]$$

Solution is statistically self-affine

Characteristic Noise

q-value	t-space	ω-space (PSDF)	Name
$q = 0$	$\frac{1}{t} \otimes n(t)$	1	White noise
$q = 1$	$\frac{1}{\sqrt{t}} \otimes n(t)$	$\frac{1}{\lvert\omega\rvert}$	Pink noise
$q = 2$	$\int^{t} n(t)dt$	$\frac{1}{\omega^2}$	Brown noise
$q > 2$	$t^{(q/2)-1} \otimes n(t)$	$\frac{1}{\lvert\omega\rvert^q}$	Black noise

Levy Statistics and the Fractional Diffusion Equation



Levy's question:

Under what circumstances does the distribution associated with a random walk of a few steps look the same as the distribution after many steps (except for scaling)?

Under what circumstances do we obtain a random walk that is *statistically self-affine*?

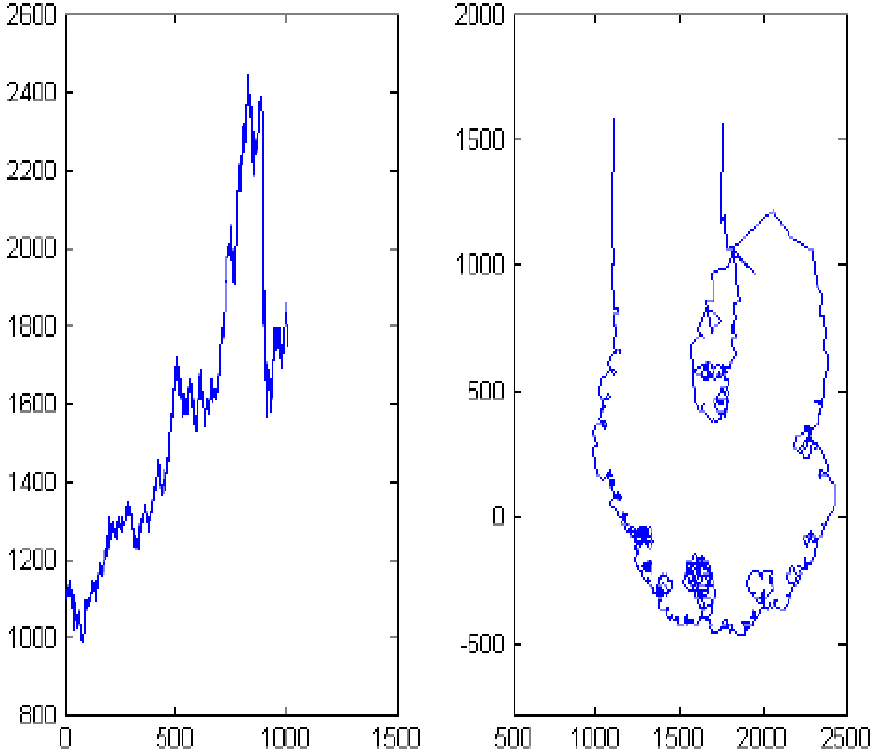
Characteristic Function and the Probability Density Function

- Levy's characteristic function:

$$P(k) = \exp(-a \mid k \mid^{\gamma}), \quad 0 < \gamma \leq 2$$

- PDF:

$$p(x) \sim \frac{1}{x^{1+\gamma}}, \quad x \to \infty$$



Evolution Equation for a Brownian Process

$$u(x, t + \tau) = u(x, t) \otimes p(x)$$

- Describes the concentration of particles that move over a distance *x* with probability ***p(x)***.

- In Fourier space this equation is

$$U(k, t + \tau) = U(k, t)P(k)$$

Fractional Diffusion Equation

- Consider the characteristic function

$$P(k) \simeq 1 - a \mid k \mid^{\gamma}$$

- We can then write

$$\frac{U(k, t+\tau) - U(k,t)}{\tau} \simeq -\frac{a}{\tau} \mid k \mid^{\gamma} U(k,t)$$

$$\sigma \frac{\partial}{\partial t} u(x,t) = \frac{\partial^{\gamma}}{\partial x^{\gamma}} u(x,t)$$

$$\tau \longrightarrow 0$$

$$\sigma = \tau / a$$

Relationship between the Levy Index and the Fourier Dimension

- Green's function is given by

$$g(|x|,\omega)=\frac{i}{2\Omega_\gamma}\exp(i\Omega_\gamma|x|)$$

$$\Omega_\gamma=i^{\frac{2}{\gamma}}(i\omega\sigma)^{\frac{1}{\gamma}}$$

- Implies the following relationship:

$$\frac{1}{\gamma}=\frac{q}{2}$$

Summary

- Random walks with a directional bias are Hurst process – *fractional Brownian motion*

- Hurst processes are a generalisation of Brownian motion and classical diffusion

$$R(t) = at^{H} \qquad \Longrightarrow \frac{\partial^2}{\partial x^2} - \sigma^q \frac{\partial^q}{\partial t^q}$$

Summary (continued)

- We have considered a model for a financial signal compounded in the fractional PDE

$$\left(\frac{\partial^2}{\partial x^2} - \sigma^q \frac{\partial^q}{\partial t^q}\right) u(x,t) = \delta(x) n(t)$$

- The solution to this equation is, for $x \rightarrow 0$

$$u(t) = \frac{1}{t^{1-q/2}} \otimes n(t)$$

which is a random fractal signal with property

$$\lambda^{q/2} \Pr[u(\lambda t)] = \Pr[u(t)]$$

In the Following Lecture…

- We shall investigate the properties of financial signals

- Consider a non-stationary model based on the operator

$$\frac{\partial^2}{\partial x^2} - \sigma^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}}$$

- Develop a moving window based algorithm to compute *q(t)* for a financial signal

Questions

+

Interval (10 Minutes)

Part II: Contents

- The Efficient Market Hypothesis
- Properties of Financial Signals
- The Fractal Market Hypothesis
- Numerical Algorithms
- Example Results
- Case Study: ABX indices
- Questions

The Efficient Market Hypothesis

The Theory of Speculation

Louis Bachelier, PhD Thesis, 1900

- Used Brownian motion to evaluate stock options
- Basis for the EMF

Efficient Market Hypothesis

Example of an EMH Model: The Black-Scholes Equation

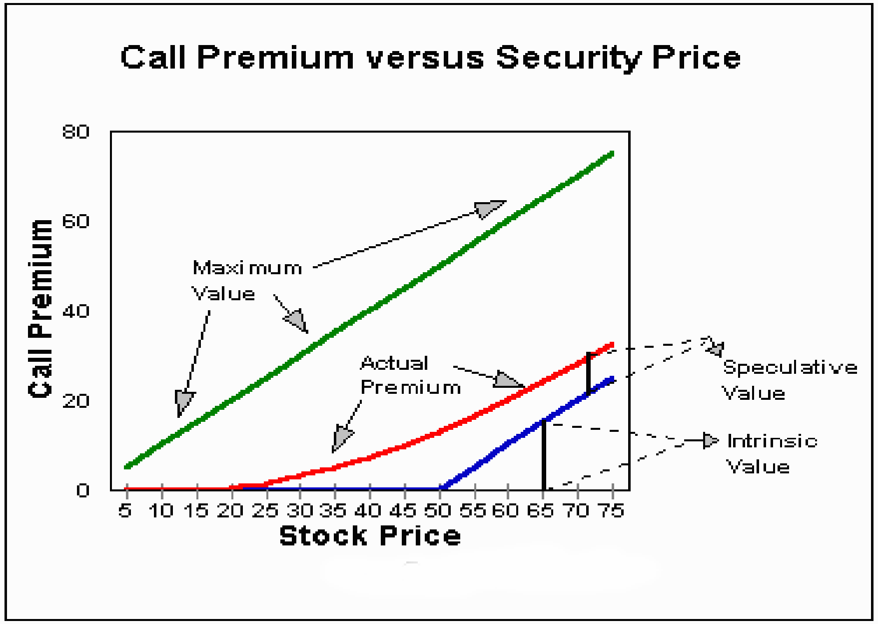
$$\frac{\partial U}{\partial t} + \frac{1}{2} v^2 x^2 \frac{\partial^2 U}{\partial x^2} + r x \frac{\partial U}{\partial x} - rU = 0$$

$U(x, t)$ - Call Premium

x - Stock Price

v - volatilty

r - risk

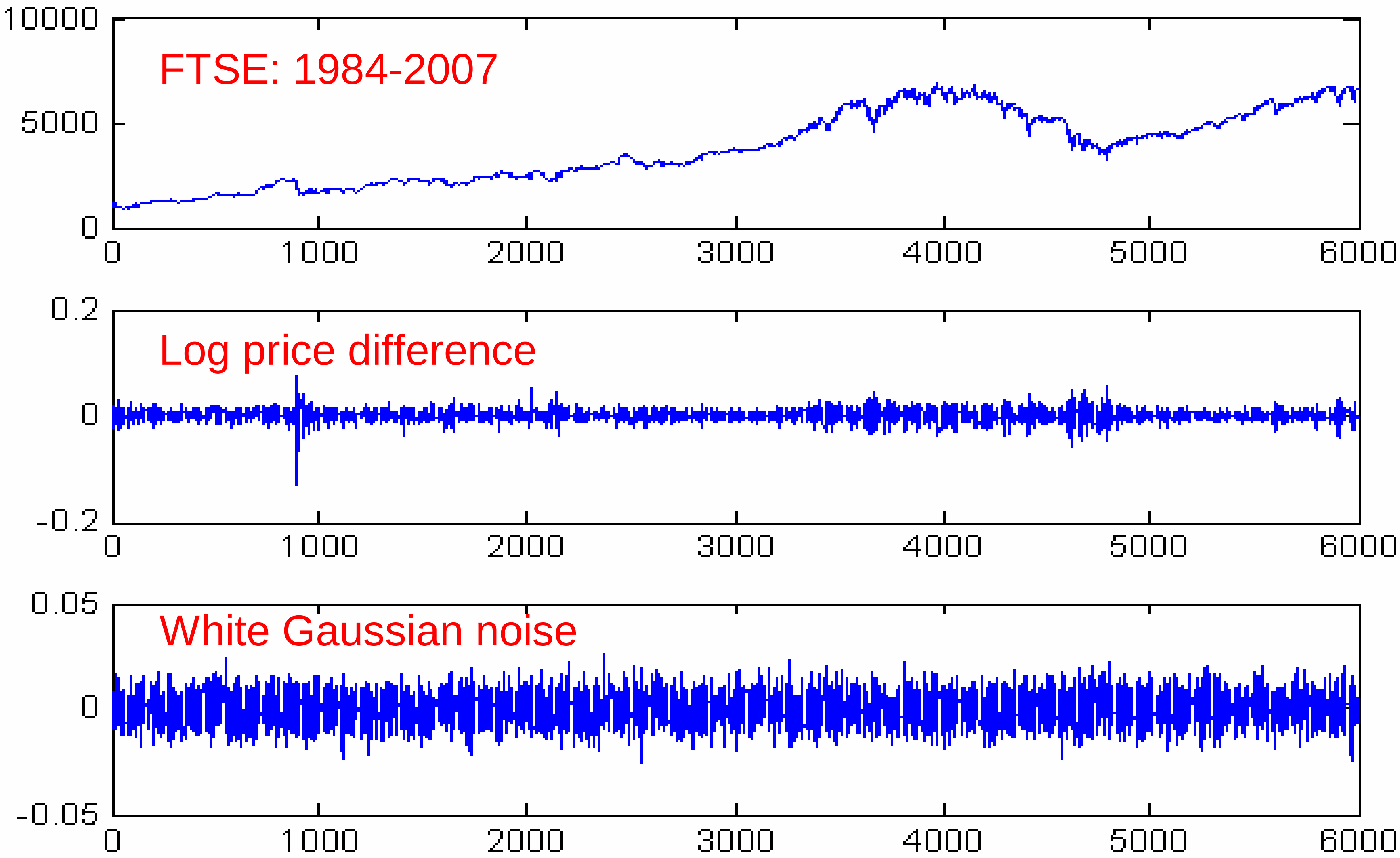


Assumes market is a stationary Gaussian process!

EMH .v. FMH: Principal Differences

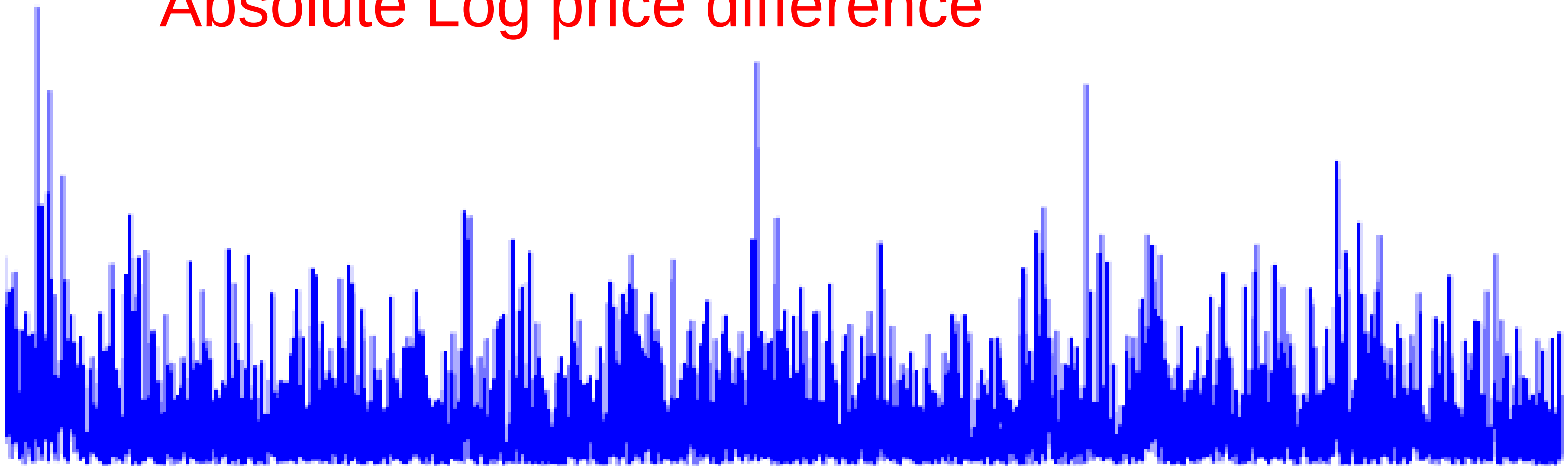
EMH	FMH
Gaussian statistics	Non-Gaussian statistics
Stationary process	Non-stationary process
Economy has no memory (no historical correlations)	Economy has memory (historical correlations exist)
No repeating patterns at any scale	Many repeating patterns at all scales, e.g. Elliot waves
Continuously stable at all scales	Possible instabilities at any scale, e.g. 'Levy Flights' and 'Black Swans'

Is an Economy Based on Stationary Gaussian Processes?

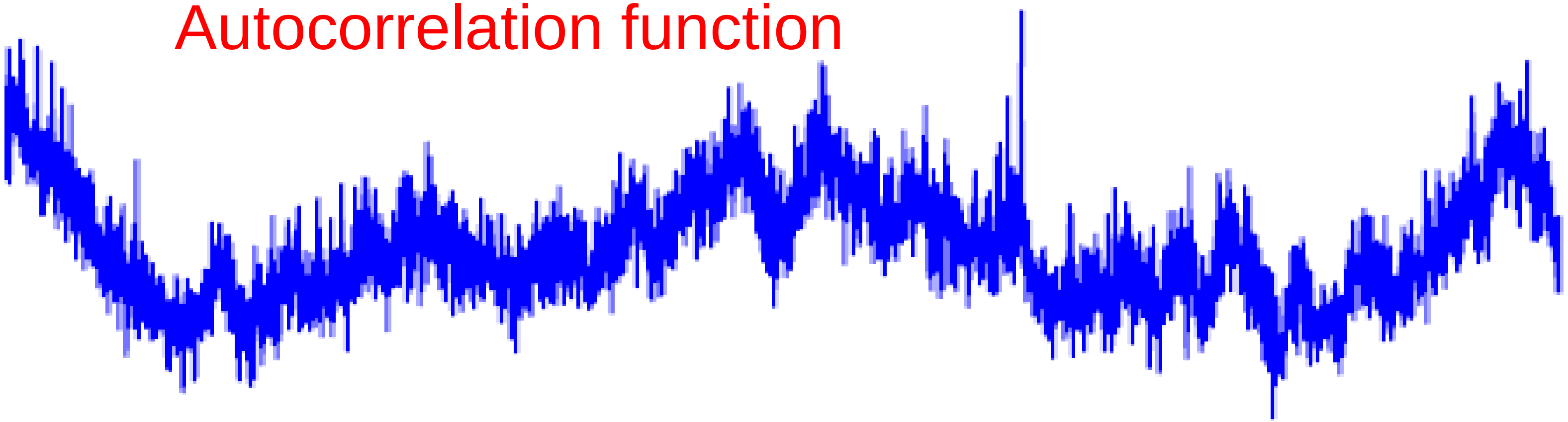


Does an Economy have Memory?

Absolute Log price difference



Autocorrelation function



Memory and Fractional Differentiation

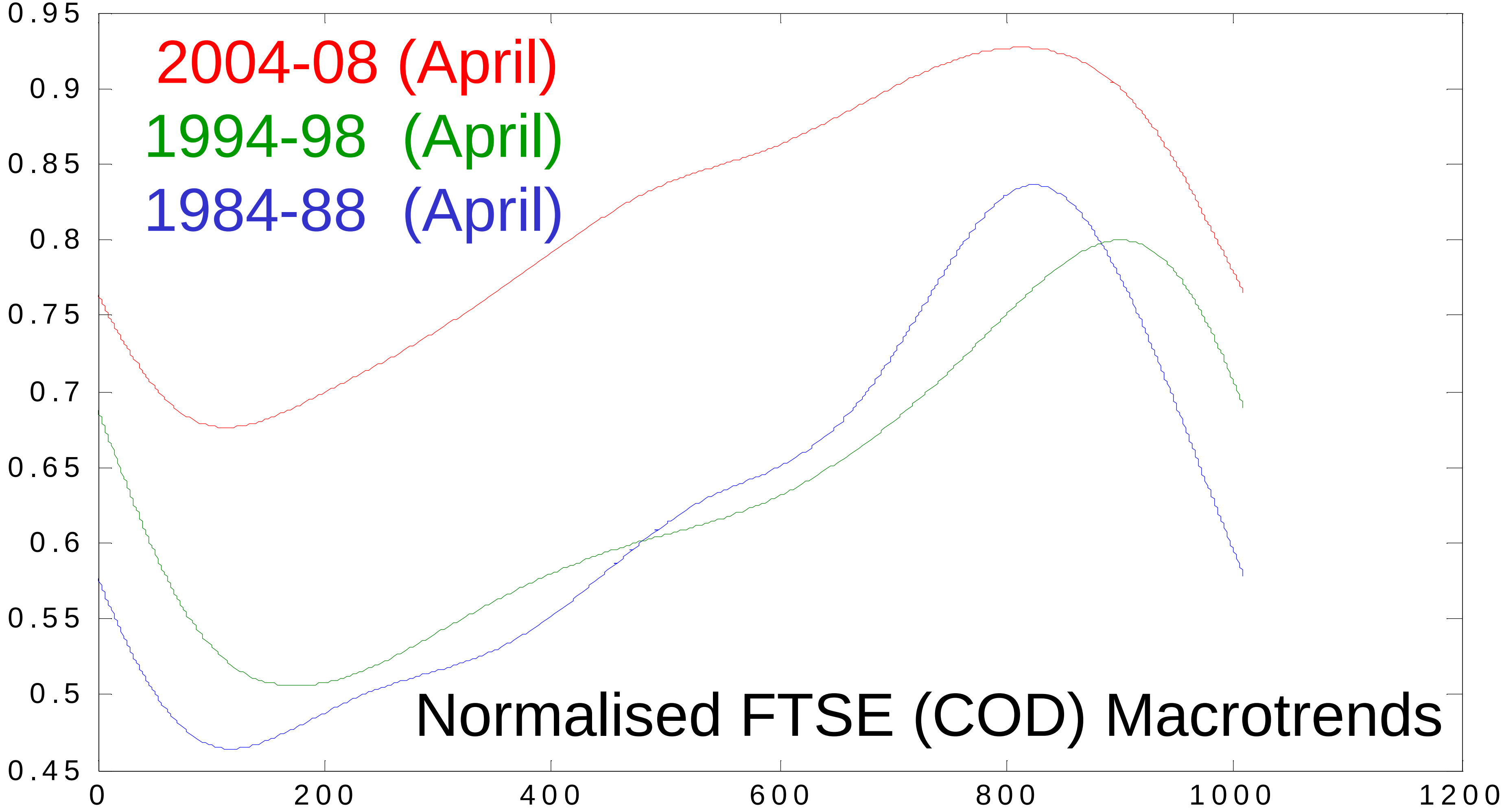
- Differentiation of a function is a localised operation; it ‘measures’ the gradient of a function at a point

- Fractional differentiation depends on the *history* of the function – its *memory*

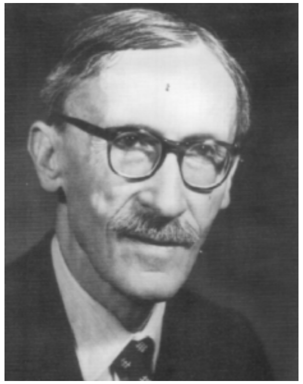
$$\hat{D}^q f(t) = \frac{d^m}{dt^m}[\hat{I}^{m-q} f(t)], \quad m - q > 0$$

$$\hat{I}^p f(t) = \frac{1}{\Gamma(p)} f(t) \otimes \frac{1}{t^{1-p}}, \quad p > 0$$

Does an Economy Have Repeating Patterns? (Elliot Waves)

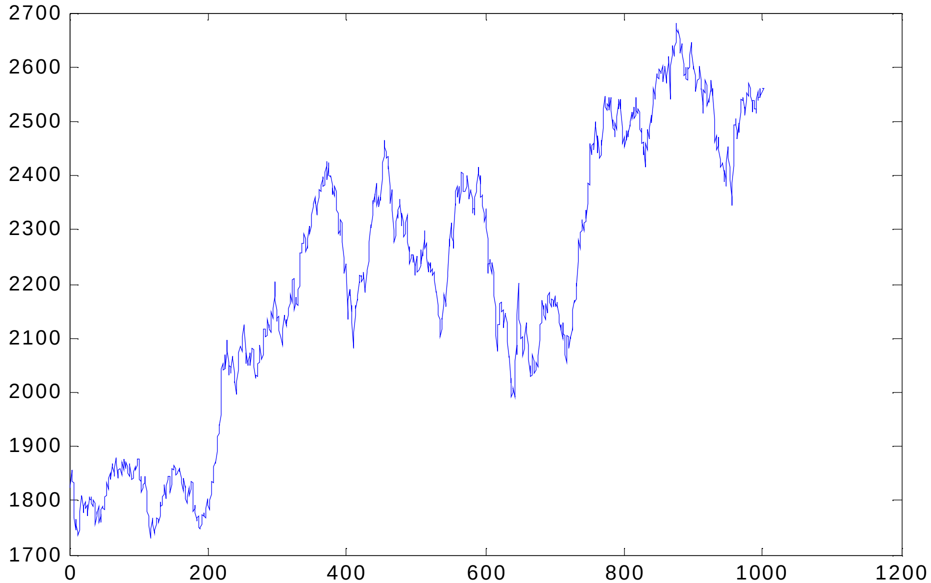


Is an Economy Continuously Stable at all Scales?



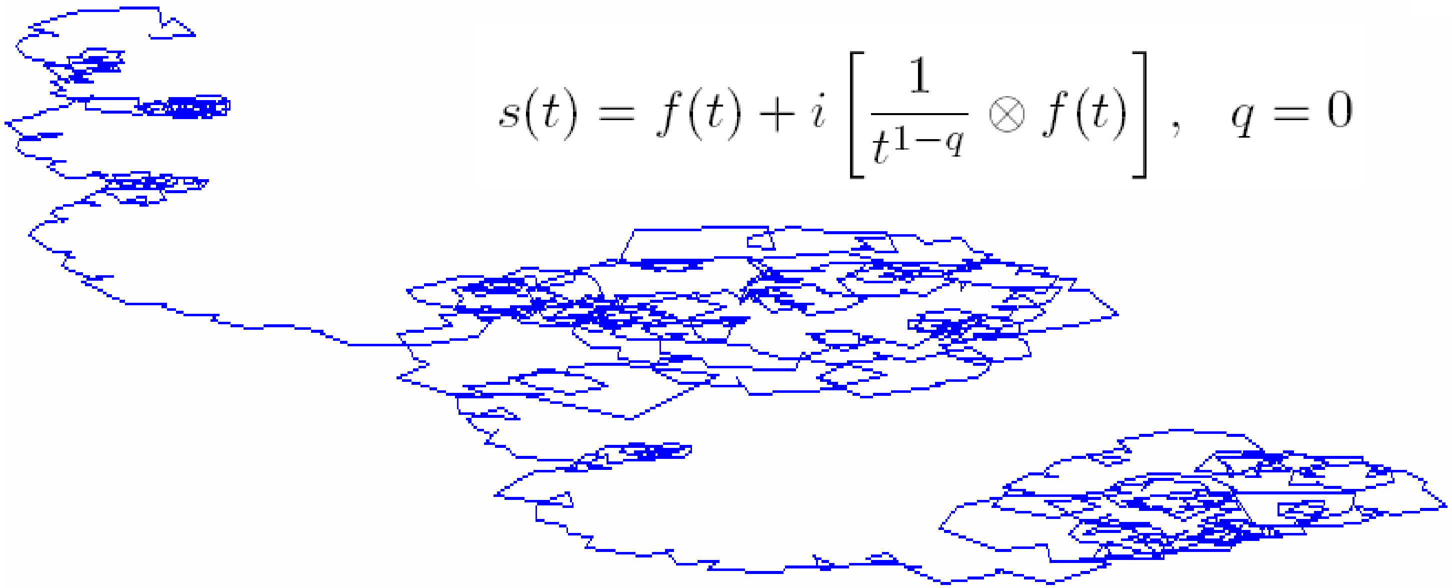
$$\Pr[f(t)] \sim \frac{1}{x^{1+p}}$$

$$0 < p < 2$$



$$\lambda^q \Pr[f(t)] = \Pr[f(\lambda t)]$$

$$F(\omega) \sim \frac{1}{(i\omega)^q}, \quad q > 0$$

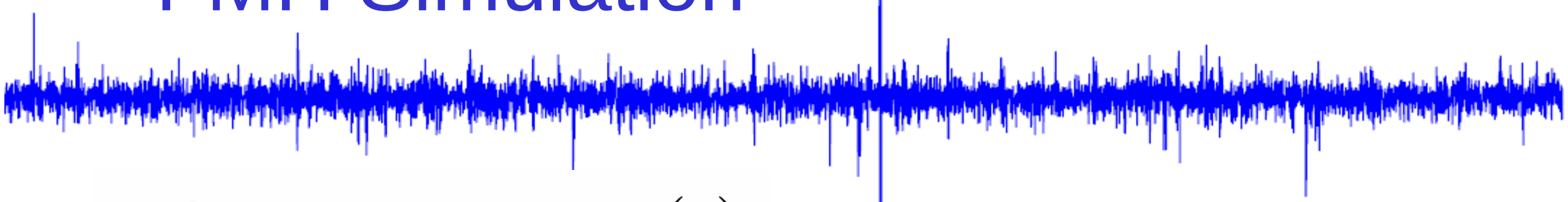


$$s(t) = f(t) + i\left[\frac{1}{t^{1-q}} \otimes f(t)\right], \quad q = 0$$

Hilbert Transform

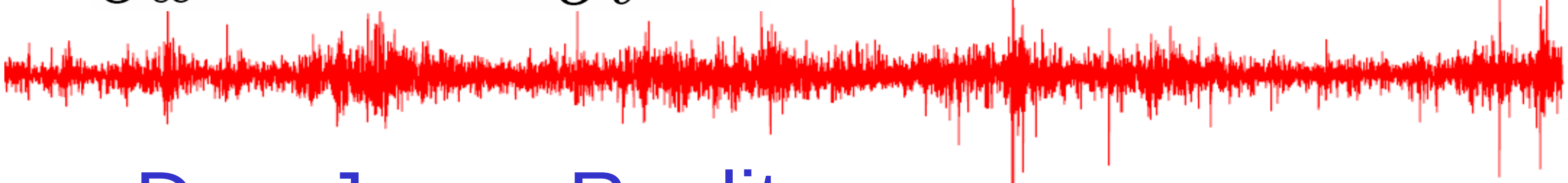
Does the FMH work?

FMH Simulation



$$\frac{\partial^2}{\partial x^2} - \sigma^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}}$$

$\Pr[q(t)] = \text{Gauss}(x)$



Dow Jones Reality

EMH

Non-stationary Signal Processing

$$\frac{\partial^2}{\partial x^2} - \sigma^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}}$$

- Requires application of ‘moving windows’ to compute *q(t)*:

  Time-frequency methods in DSP

- Hypothesis:

  A change in *q(t)* precedes a change in a macroeconomic index (e.g. FTSE)

Computing *q(t)*

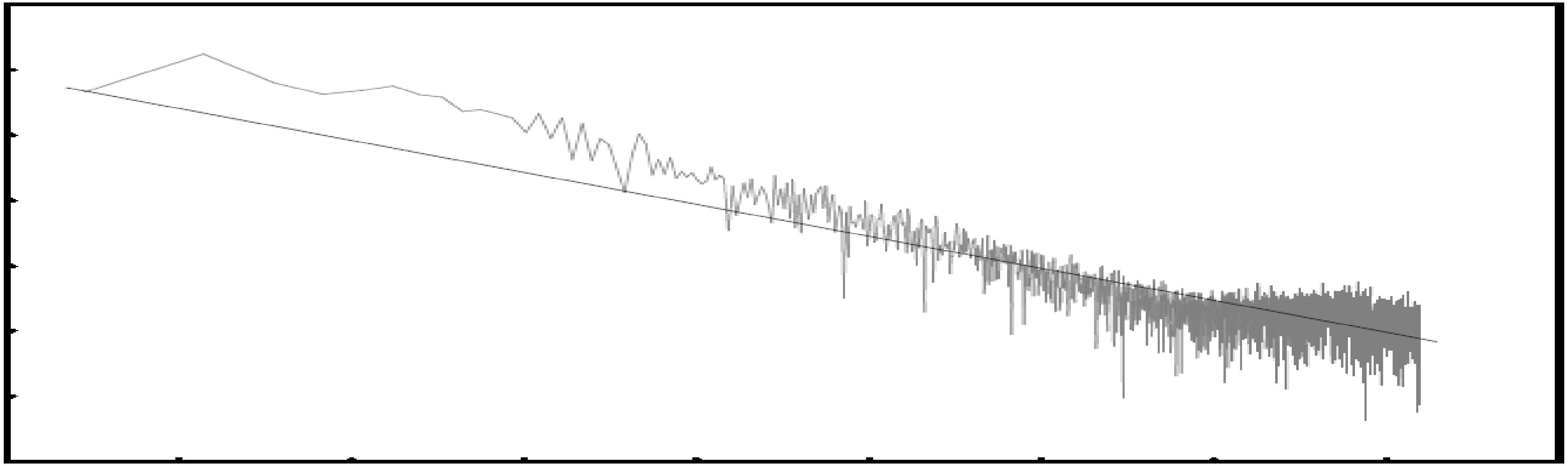
- Applying a moving window to the signal (user defined)

- For each window, assume a stationary process where the signal is given by

$$u(t) = \frac{1}{t^{1-q/2}} \otimes n(t), \quad q > 0$$

Power Spectrum Method

$$U(\omega) = \frac{N(\omega)}{(i\omega)^{q/2}} \qquad \hat{P}(\omega) = \frac{c}{\omega^q}$$



$$\ln \hat{P}(\omega) = C + q \ln \omega$$

Data Fitting Methods

- Least Squares Method: Minimise

$$e(q,C) = \| \ln P(\omega) - \ln \hat{P}(\omega, q, C) \|_2^2$$

- Orthogonal Linear Regression (OLR)

- Note: DC level is omitted from input as it measures the scale of the signal and not its fractal dimension

Principal Algorithm

Step 1: Read data (financial time series) from file into operating array $a[i], i = 1, 2, ..., N$.

Step 2: Set length $L < N$ of moving window w to be used.

Step 3: For $j = 1$ assign $L + j - 1$ elements of $a[i]$ to array $w[i], i = 1, 2, ..., L$.

Step 4: Compute the power spectrum $P[i]$ of $w[i]$ using a Discrete Fourier Transform (DFT).

Principal Algorithm (continued)

Step 5: Compute the logarithm of the spectrum excluding the DC, i.e. compute $\log(P[i]) \forall i \in [2, L/2]$.

Step 6: Compute $q[j]$ using the OLR algorithm.

Step 7: For $j = j + 1$ repeat Step 3 - Step 5 stopping when $j = N - L$.

Step 8: Write the signal $q[j]$ to file for further analysis and post processing.

The Wavelet Transform

$$\mathcal{W}[f(t)] = F_L(t) = \int\limits_{-\infty}^{\infty} f(\tau) w_L(\tau, t) d\tau$$

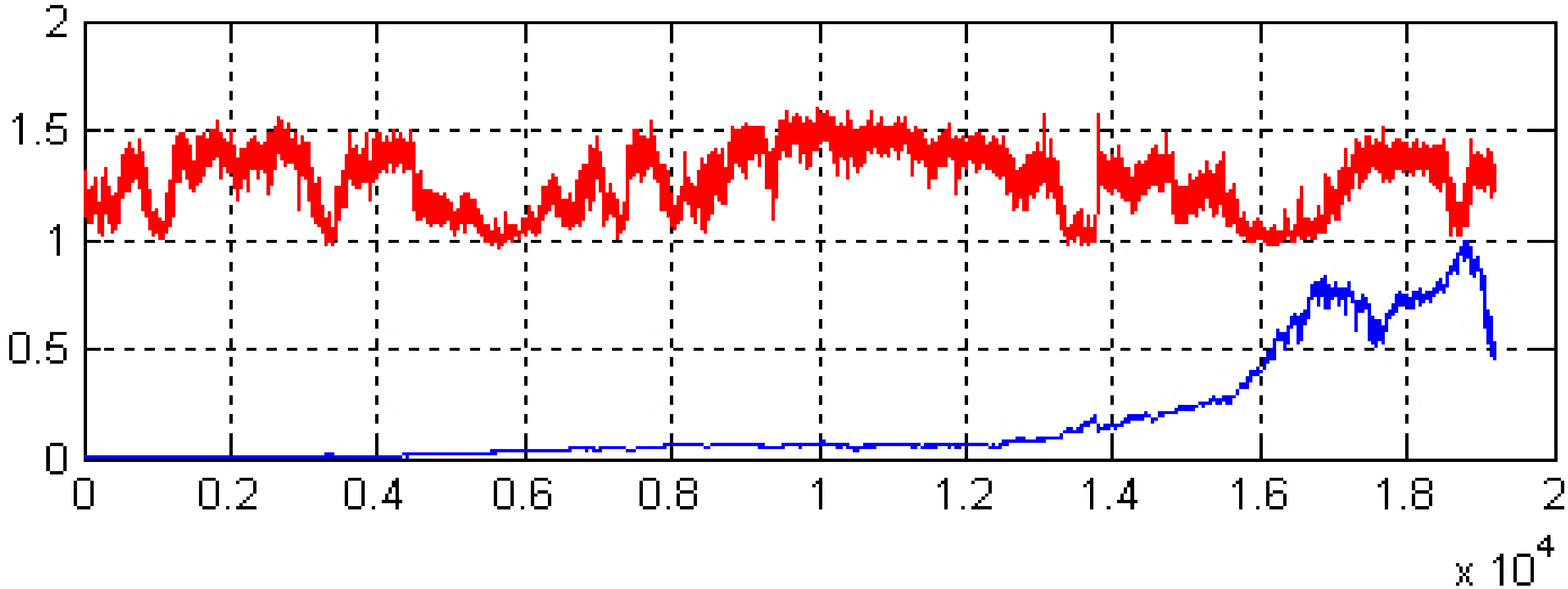
$$w_L(\tau, t) = \frac{1}{\sqrt{L}} w\left(\frac{\tau - t}{L}\right), \quad L > 0.$$

FMH Wavelet

$$F_L(t) = w_L(t) \otimes f(t), \quad L > 0$$

$$w_1(t, \tau) = \frac{1}{t^{1-q(\tau)/2}}$$

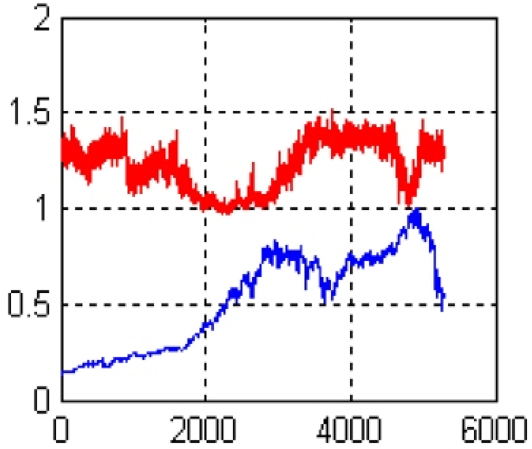
Example Result

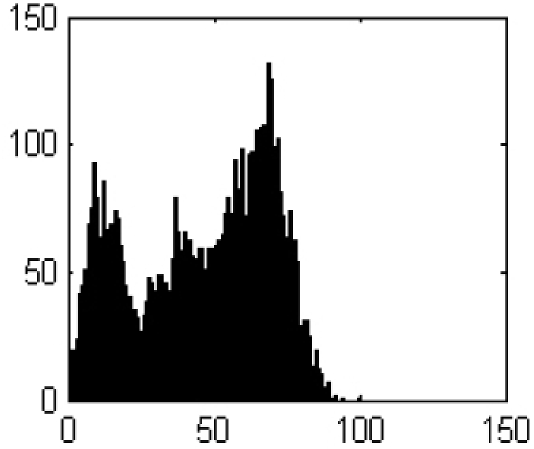


Dow Jones Close-of-Day data from 02-11-1932 to 25-03-2009.

Dow Jones data (blue – after normalization) and *q(t)* (red) computed using a window of size 1024.

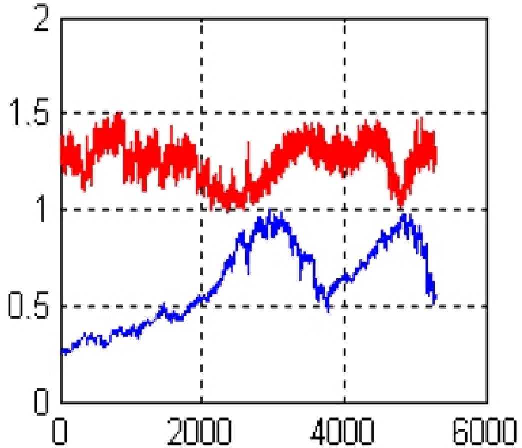
FTSE . V . DJ

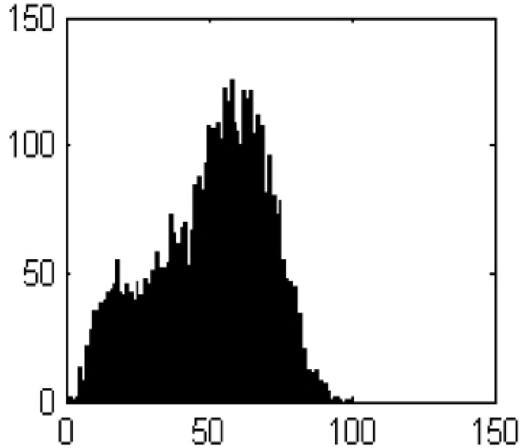




FTSE Close-of-Day data from 25-04-1988 to 20-03-2009.

100 bin Histogram of *q*





DJ Close-of-Day data from 25-04-1988 to 20-03-2009.

100 bin Histogram of *q*

Statistical Properties of *q(t)*

Statistical Parameter	$q(t)$-FTSE	$q(t)$-DJ
Minimum Value	0.9876	0.9752
Maximum value	1.5067	1.5154
Range	0.5190	0.5402
Mean	1.2482	1.2218
Median	1.2639	1.2452
Standard Deviation	0.1017	0.1269
Variance	0.0104	0.0161
Skew	-0.4080	-0.2881
Kertosis	2.3745	1.8233
Composite NormalityN	Reject	Reject

Post-processing: Filtering *q(t)*

- By low pass filtering *q(t)* macrotrends can be detected.

- Imperative that a filter is used that:

  - *guarantees smoothness*

  - *shape preserving*

  - *locally data consistent*

- Requires application of a

  Variation Diminishing Smoothing Kernel (VDSK)

Gaussian VDSK

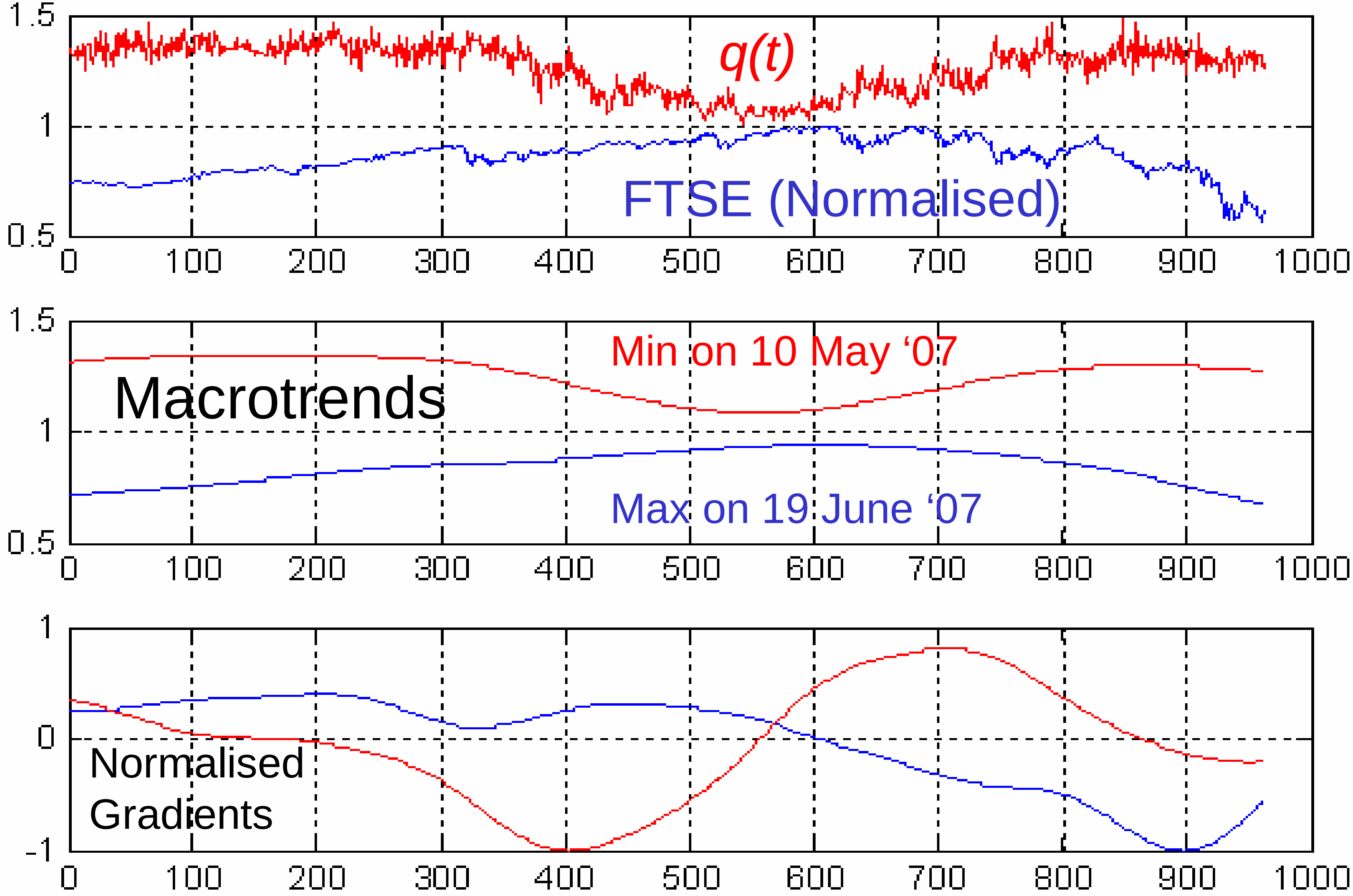
- Gaussian low-pass filter given by

$$\exp(-\beta\omega^2)$$

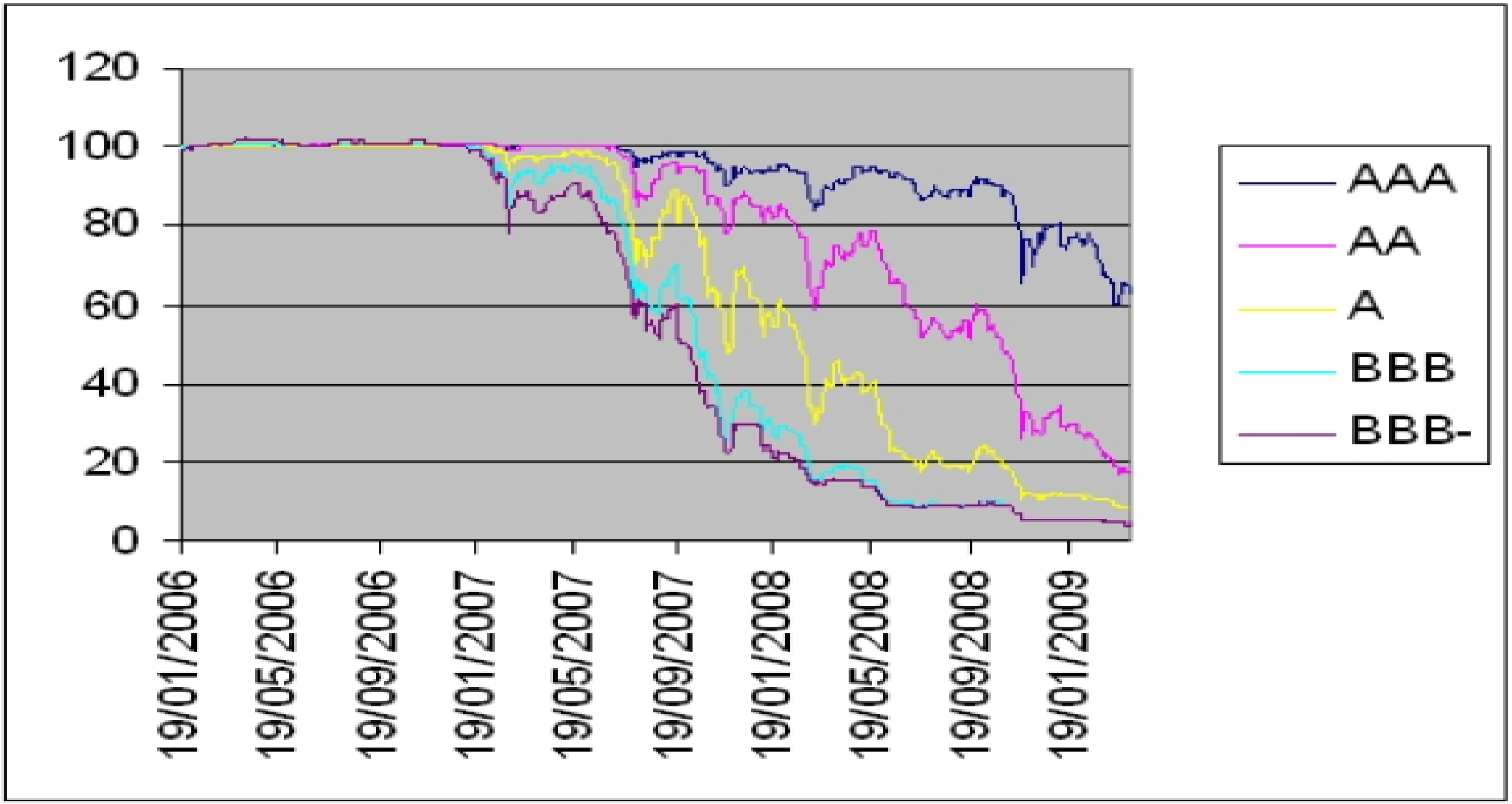
- b determines the detail observed in *q(t)*

- Filtering must be based on *end-point-extension* method in order to preserve consistency of the filtered data, i.e. *eliminate wrapping*

Example: FTSE (Close-of-Day) 19 March 2004 – 26 November 2008



Case Study: ABX Index Analysis

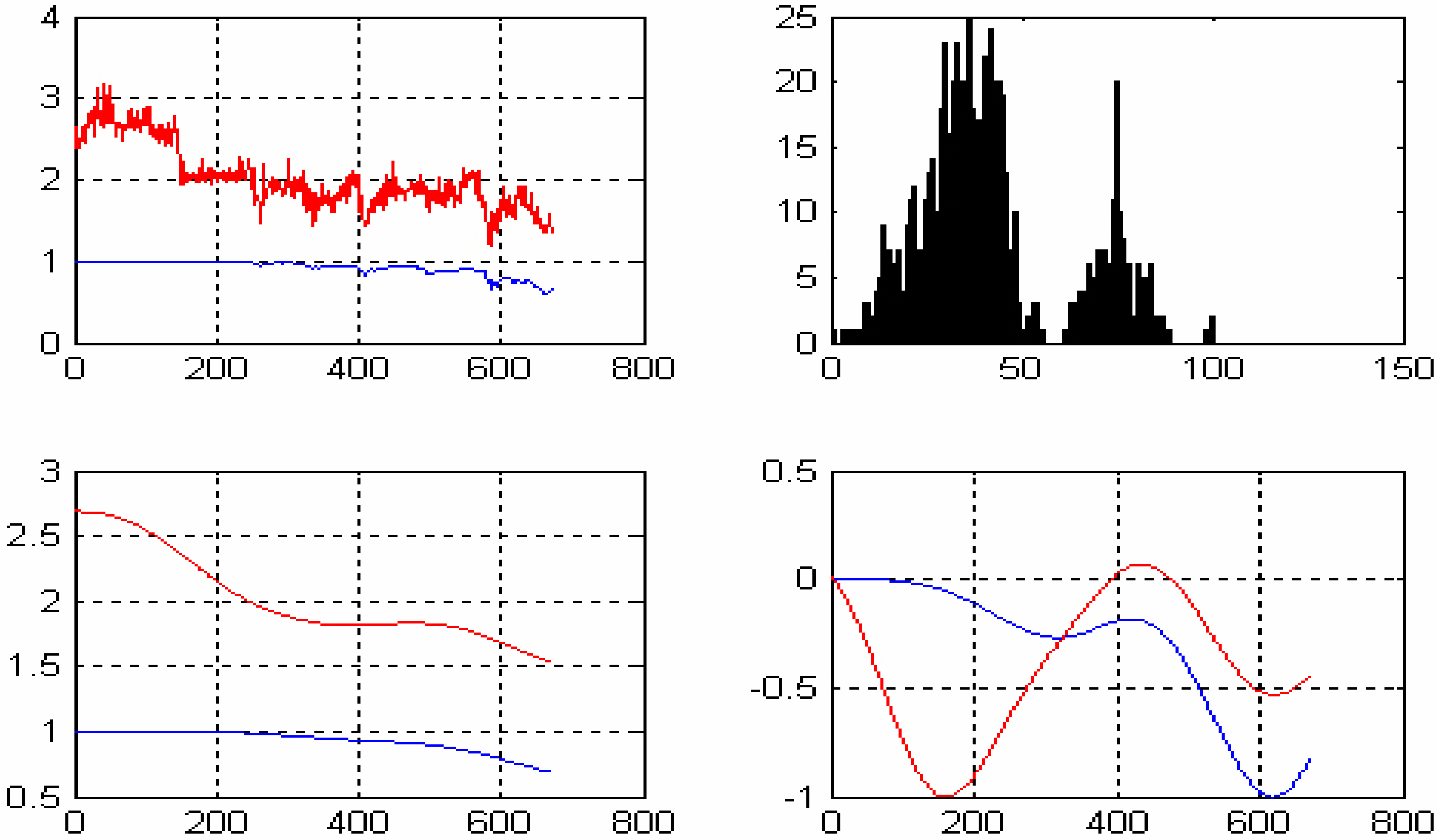


Grades for the ABX Indices from 19 January 2006 to 2 April 2009 based on Close-of-Day prices.

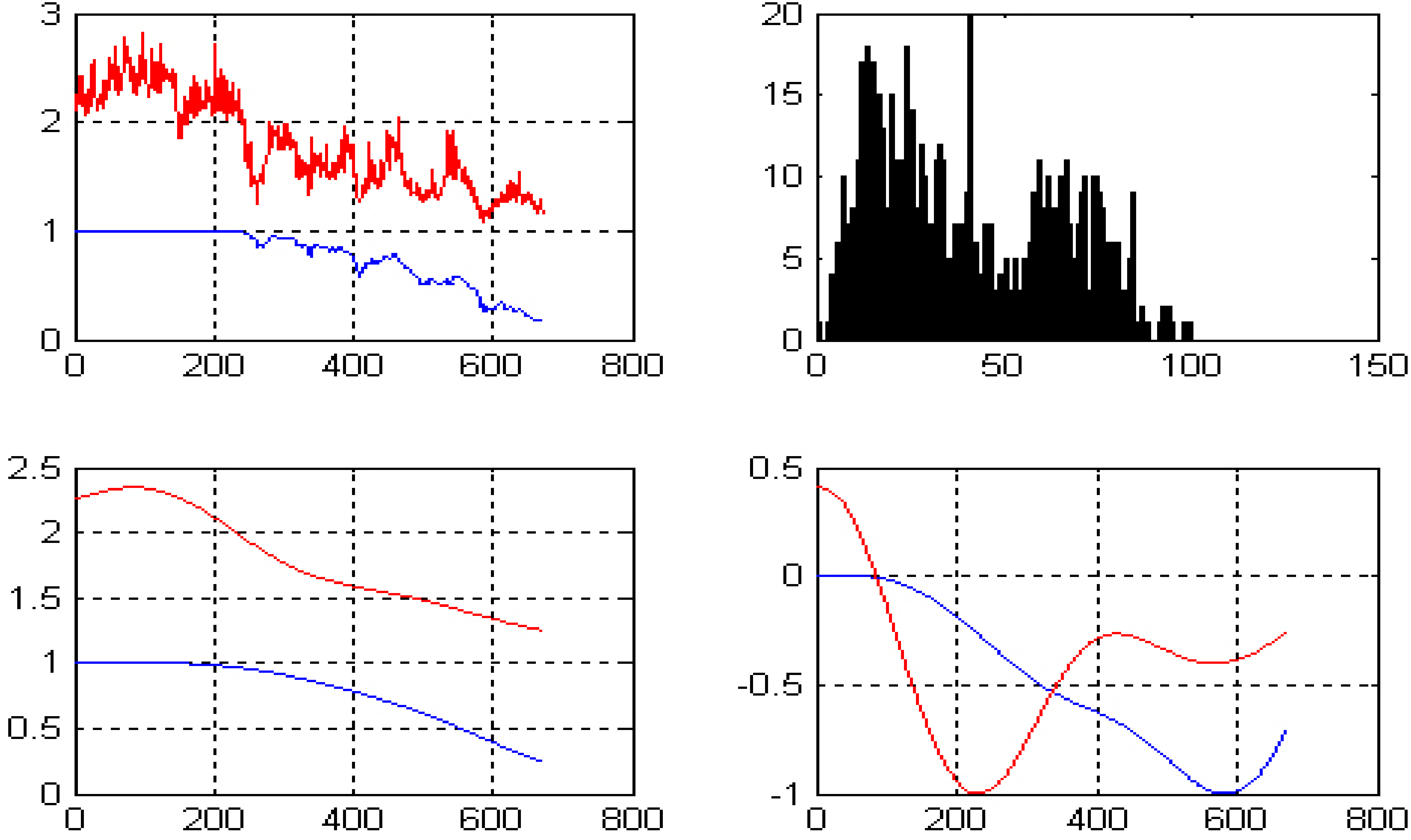
What is an ABX Index ?

- The index is based on a basket of Credit Default Swap (CDS) contracts for the sub-prime housing equity sector.
- Credit Default Swaps operate as a type of insurance policy for banks or other holders of bad mortgages.
- If the mortgage goes bad, then the seller of the CDS must pay the bank for the lost mortgage payments.
- Alternatively, if the mortgage stays good then the seller makes a lot of money.
- The riskier the bundle of mortgages the lower the rating.

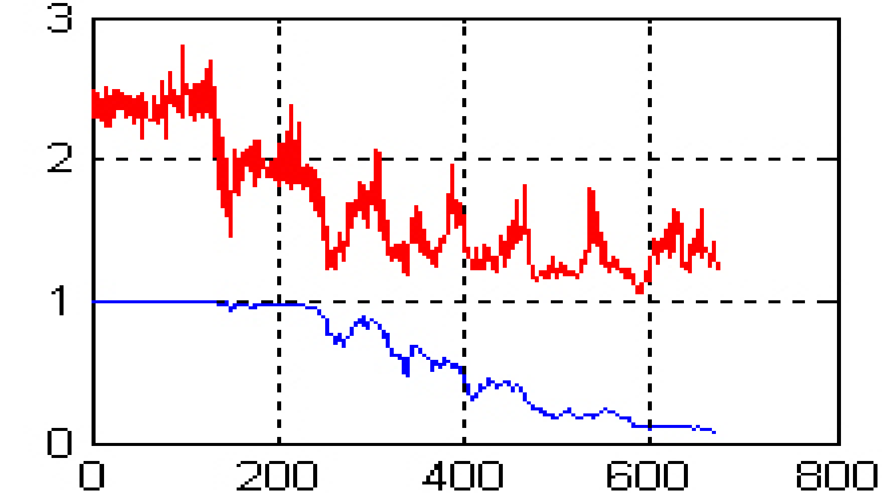
AAA: 24-07-2006 - 02-04-2009

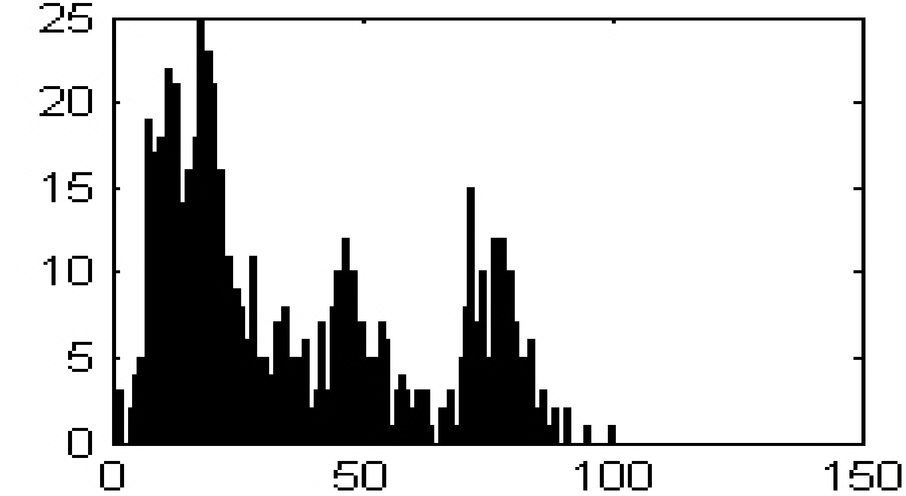


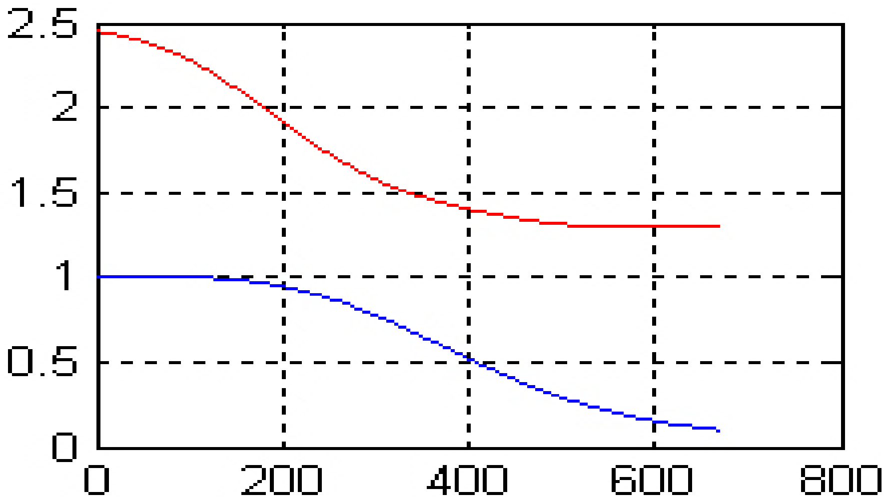
AA: 24-07-2006 - 02-04-2009

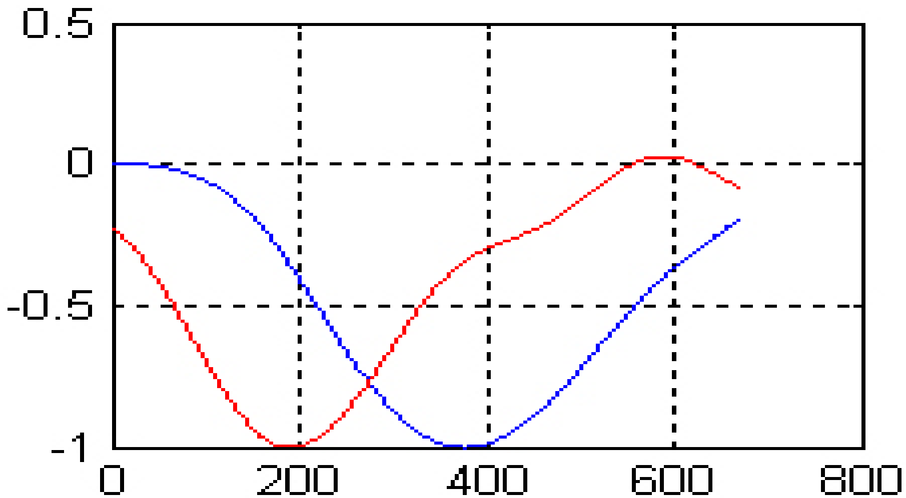


A: 24-07-2006 - 02-04-2009

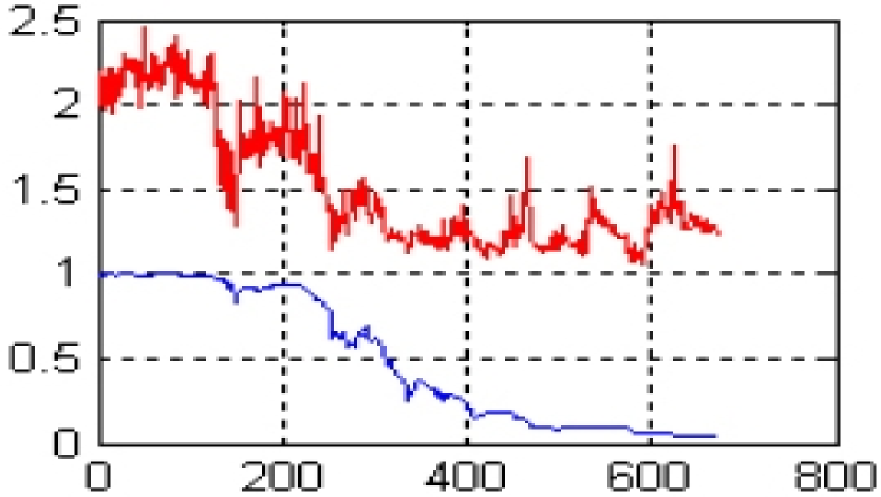


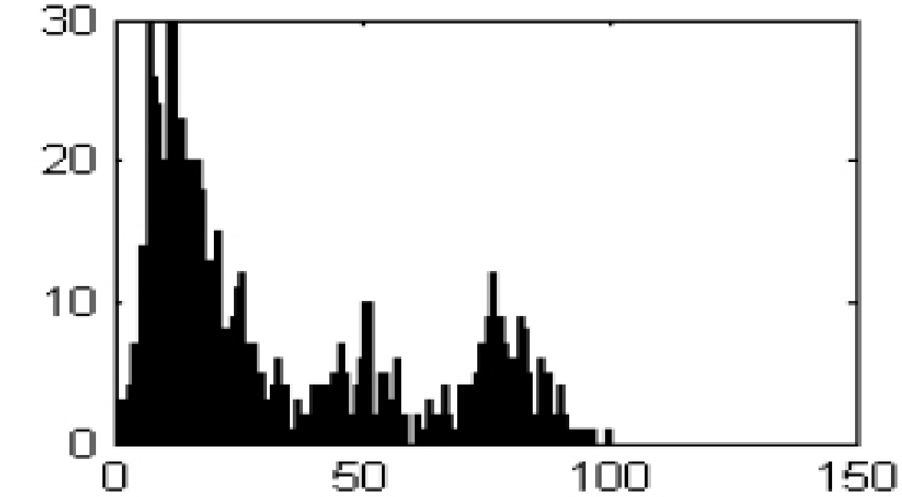


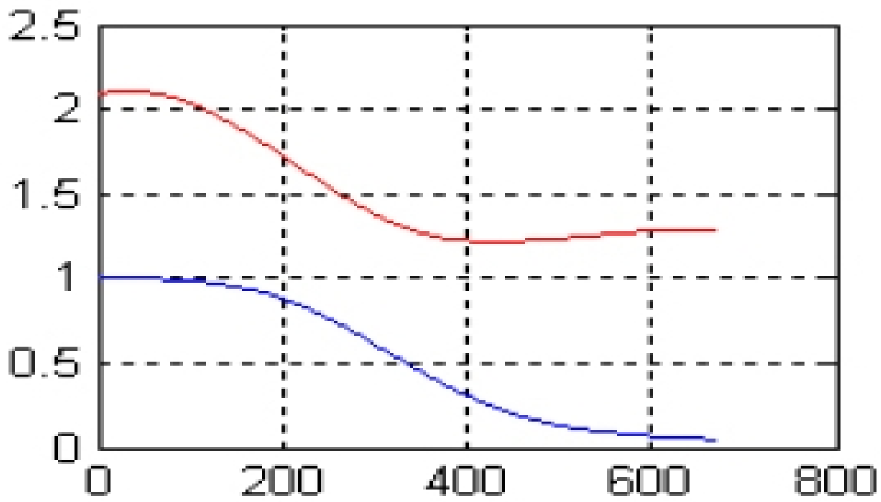


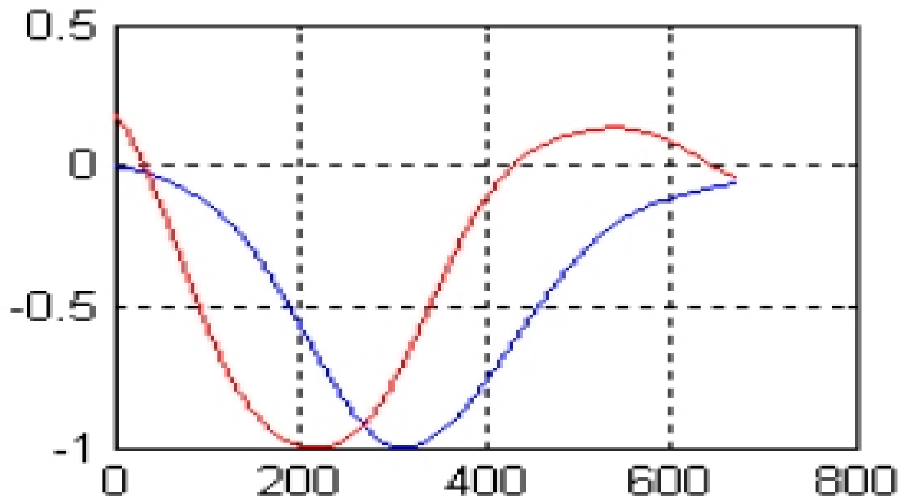


BBB: 24-07-2006 - 02-04-2009

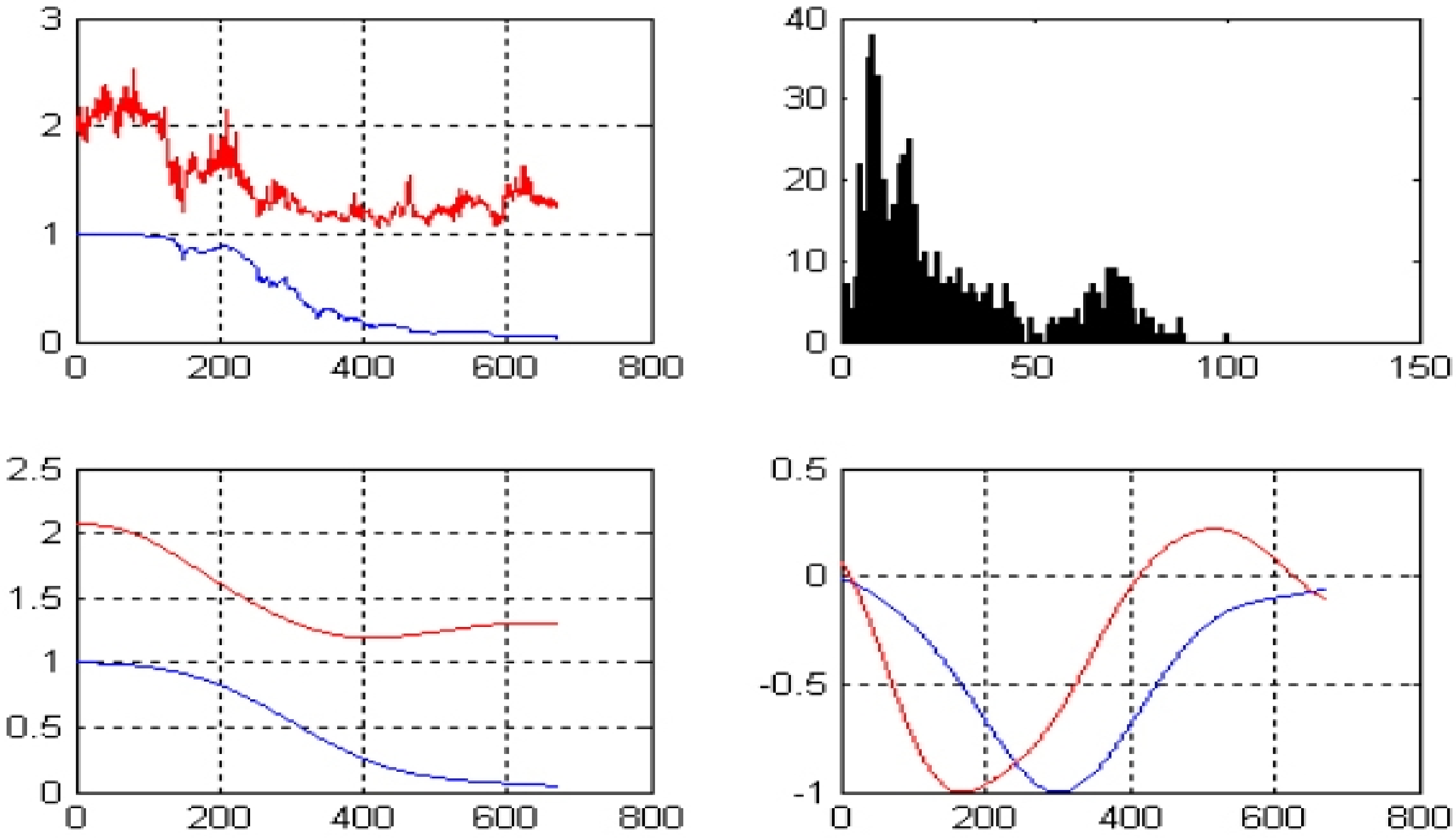








BBB-: 24-07-2006 - 02-04-2009

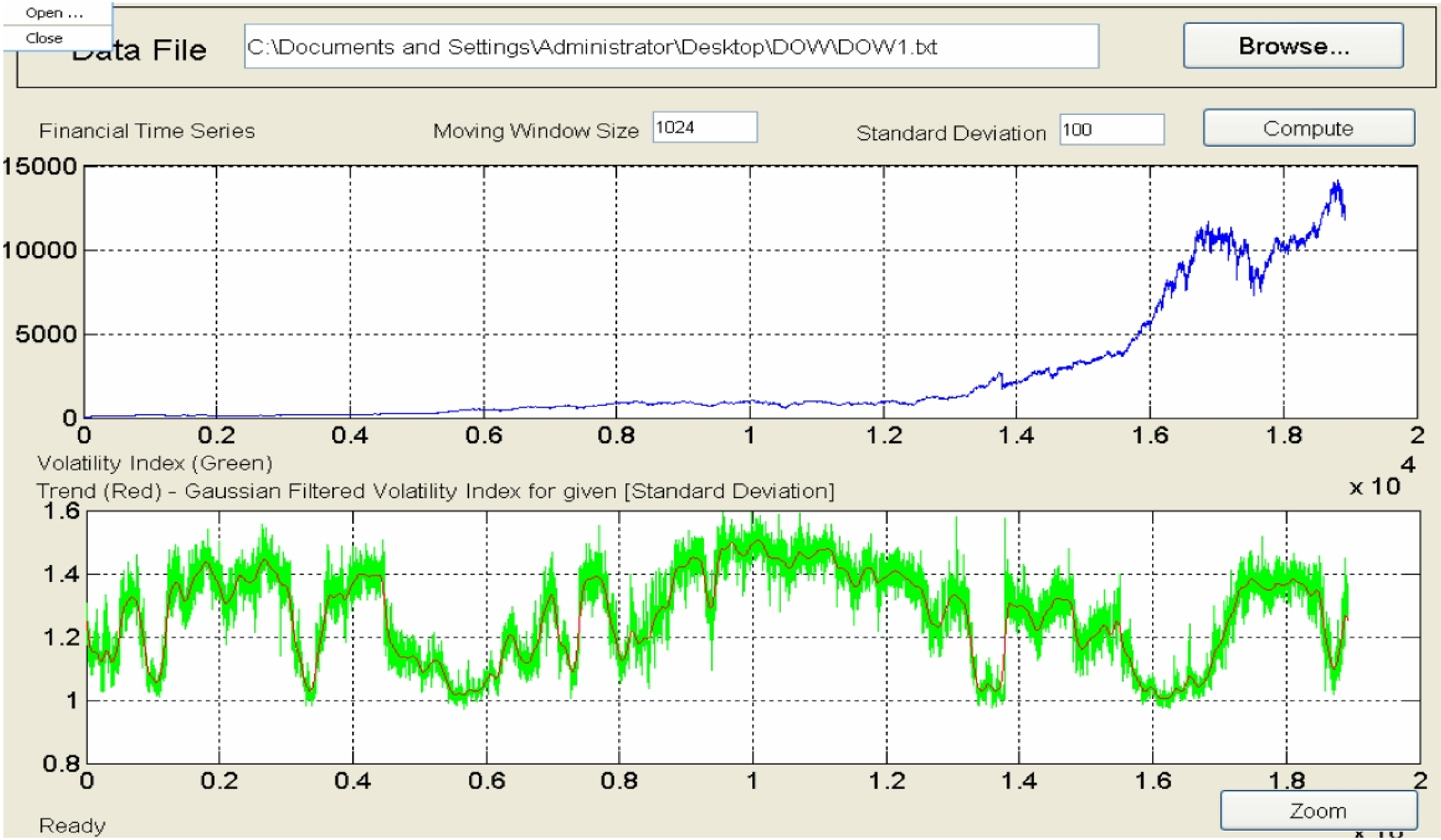


Statistical Characteristics of *q(t)* for ABX Indices

Statistical Parameter	AAA	AA	A	BBB	BBB-
Min.	1.1834	1.0752	1.0522	1.0610	1.0646
Max.	3.1637	2.8250	2.7941	2.4476	2.5371
Range	1.9803	1.7499	1.7420	1.3867	1.4726
Mean	2.0113	1.7869	1.6663	1.5141	1.4722
Median	1.9254	1.7001	1.4923	1.3425	1.3243
SD	0.3928	0.4244	0.4384	0.3746	0.3476
Variance	0.1543	0.1801	0.1922	0.1404	0.1208
Skew	0.7173	0.3397	0.6614	0.8359	1.0345
Kertosis	2.7117	1.8479	2.0809	2.2480	2.7467
CN	Reject	Reject	Reject	Reject	Reject

Demonstration Software

http://eleceng.dit.ie/arg/downloads/FMH.zip



What Went Wrong?
The Human Factor

“I am in blood,
stepp’d in so far that,
should I wade no more,
returning were as tedious as go o’er.”

Replace the word ‘blood’ for ‘debt’ and the trend was set irrespective of the Hypothesis

Summary

- Market dynamics are fractional dynamics

- Many economic signals are non-stationary random scaling fractals

- FMH operator:

$$\frac{\partial^2}{\partial x^2} - \sigma^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}}$$

- FMH Hypothesis:

A change in q(t) precedes a change in a macroeconomic index

Summary (Continued)

- *q(t)* is computed using the OLR method and a moving window

- *q(t)* is filtered using a Gaussian VDSK to reveal macrotrends

- *q(t)* provides a measure of

Bear .v. Bull

Open Problems

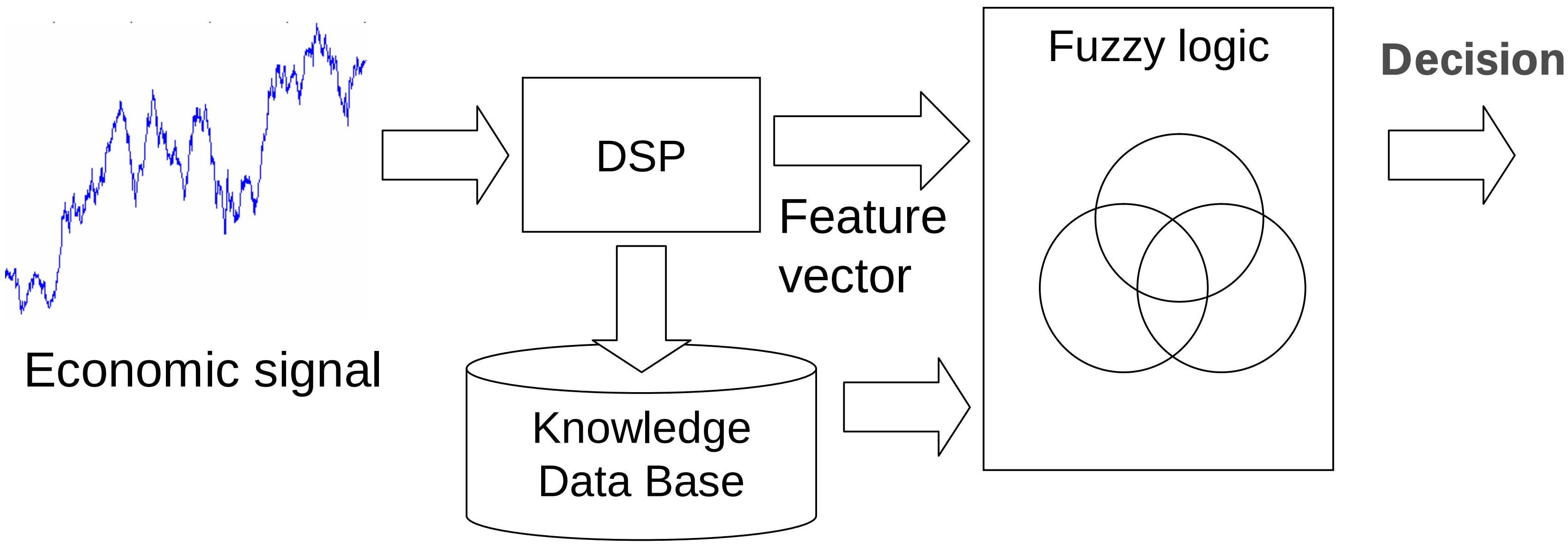
- What is the best way to interpret *q(t),* i.e. how should *q(t)* be post-processed to best 'reflect' the FMH:
  - statistical interpretation of Pr[*q(t)*] ?
  - Bayesian analysis?

- What other fractal measures could be used ?

- What is the effect of considering a multi-dimensional model with FMH operator:

$$\nabla^2 - \sigma^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}}$$

Research Project Proposal 1: Multi-Fractal Analysis



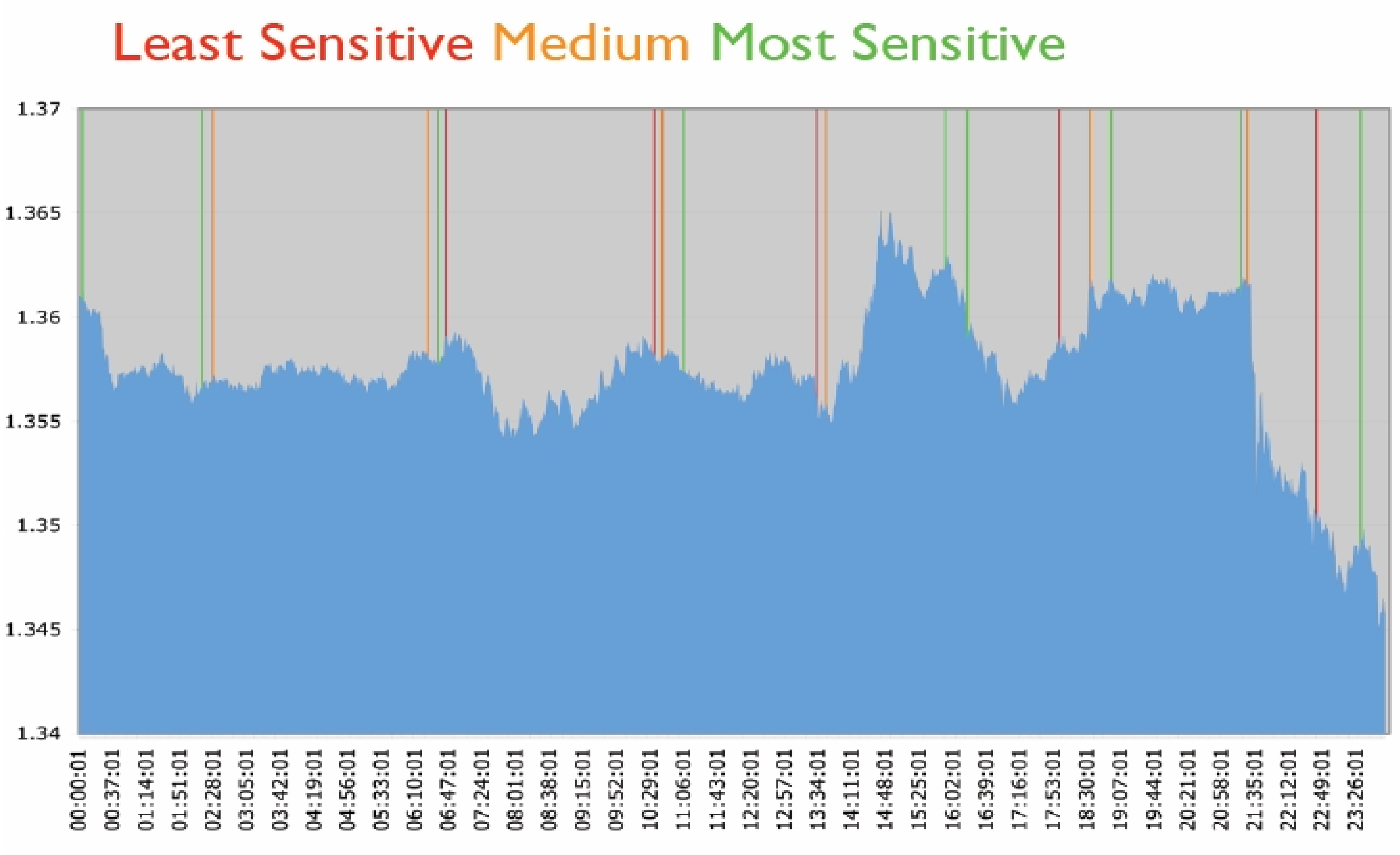
Feature vector to include multi-fractal parameters *q*=1,2,…

$$D_q = \frac{1}{(q-1)} \lim_{\delta \to 0} \frac{\ln\left(\sum_i p_i^q\right)}{\ln \delta} \qquad p_k = N_k / N$$

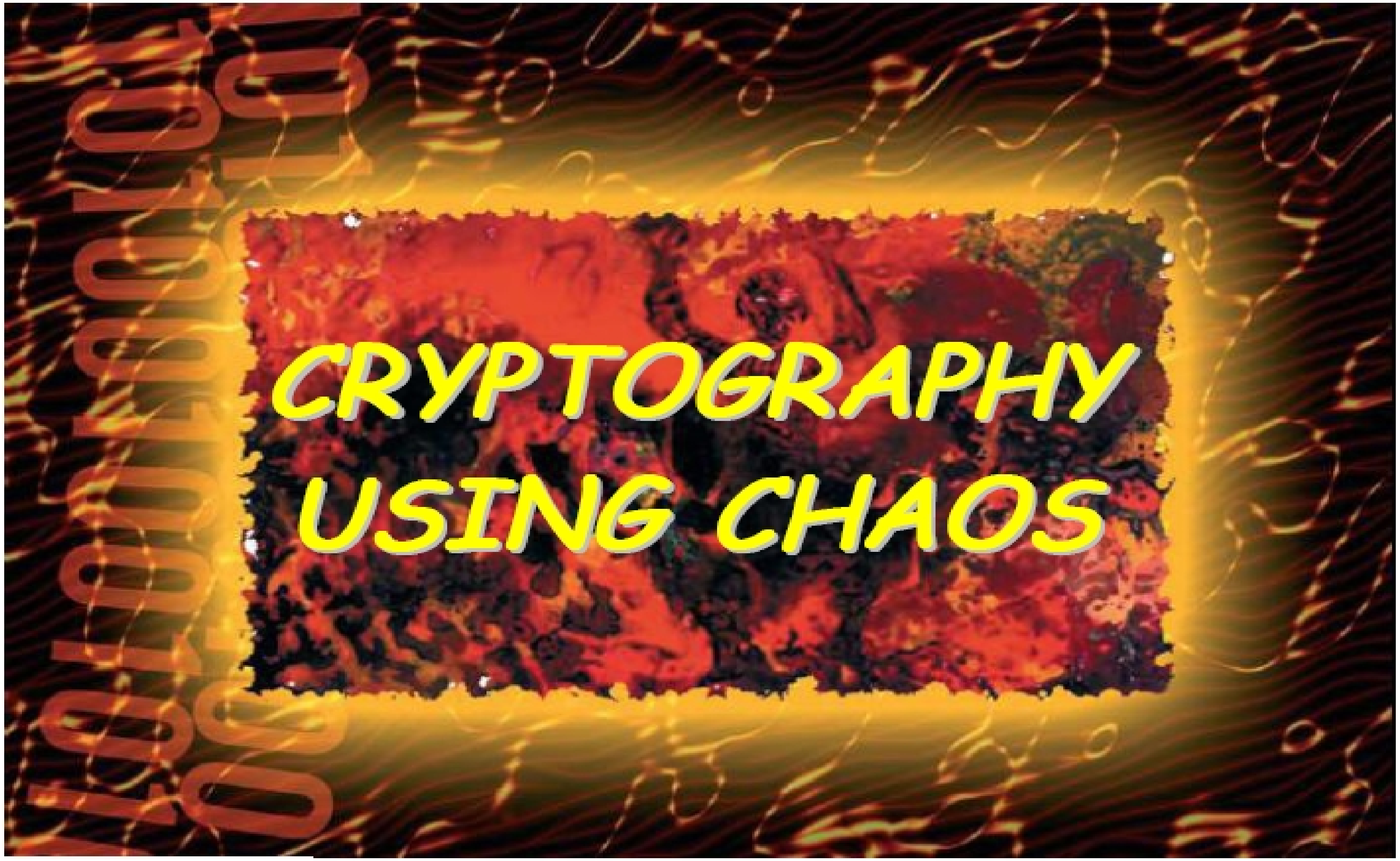
http://www.tradingsolutions.com

Research Project Proposal 2: Currency Exchange Rate Analysis

http://www.augustus.co.uk/



Q & A



Wednesday 10th March, 2008, 10:00 - 13:00

Institute of Heat Engineering,

Faculty of the Power and Aeronautical Engineering,

21/25 Nowowiejska St. Room TC 105 (first floor).

http://itc.itc.pw.edu.pl/index_an.html