Families of Linear Equations

J.W. Bruce

Let $\mathbb{K}$ denote the real or complex numbers and let A be the germ of a smooth family of $\mathbb{K}$-matrices, that is $A : \mathbb{K}^n, 0 \to M(p, q)$ is a smooth map-germ to the space of $p \times q$ matrices. Suppose given a smooth germ $f : \mathbb{K}^n, 0 \to \mathbb{K}^p$, and suppose further that for each x near 0 the linear equation $A(x)u = f(x)$ has a solution u_x. We seek conditions on A which ensure that this 'pointwise solution' ensures a smooth solution, that is a smooth germ $u : \mathbb{K}^n, 0 \to \mathbb{K}^q$ with $A(x)u(x) = f(x)$ for all x near 0; we refer to such a matrix family as being *solvable*. There is a very elegant result due to John Mather, from around 1970, which shows that any germ $A : \mathbb{K}^n, 0 \to M(p, q)$ transverse to the stratification of the matrices by rank is solvable. Since 'most' mappings will satisfy this condition 'most' families of matrices are solvable. In this paper we consider the following question: for a fixed (n, p, q) and choice of field $\mathbb{K}$ what is the codimension of the set of matrices which are not solvable? Suppose first that $\mathbb{K} = \mathbb{C}$. We show that when $n \geq p - q + 2, n \geq 2$ then the non-solvable matrices are of 'infinite co-dimension'. In particular this is true whenever $p \geq q$ and we give a geometric proof in this case. We establish similar results in the complex case for (square) symmetric and skew-symmetric matrices. When $n = 1$ or $n \geq 2$ and $n + p - q \leq 1$ non-solvable equations arise in finite co-dimension and we find some estimates for the upper bound on this codimension, noting that it is 1 when $n = q - p + 1$. When $\mathbb{K} = \mathbb{R}$ nonsolvable equations are of finite codimension for all n, p, q and we obtain some crude upper bounds for that codimension.