SOLVABILITY OF IMPLICIT HAMILTONIAN SYSTEMS

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The main theme of the lectures is the solvability problem of implicit hamiltonian systems and generalized hamiltonian systems introduced by P. A. M. Dirac. The lectures are based on my recent joint researches with Prof. S. Janeczko.

We begin with more general ordinary differential equations called implicit differential equations. Let TM be the tangent bundle of a smooth manifold M. We regard a subset S of TM as an ordinary differential equation and call it an implicit differential equation. A solution of an implicit differential equation S is a smooth curve f from an interval (a, b) into M such that (f(t), df/dt(t)) is in S for all t in (a, b). If there exists a family of solutions smoothly depending on initial conditions which covers S, we say that S is smoothly solvable.

Since solvability of differential equations is a local property, we consider the case where M is an euclidean space. Let's consider the euclidean 2n space E(2n) as a symplectic manifold endowed with the Darboux 2-form. Then its tangent bundle TE(2n) is naturally endowed with a symplectic structure, induced from that of the cotangent bundle of E(2n) by the isomorphism between the tangent bundle and the cotangent bundle defined through the Darboux form.

Throughout the lectures, we are interested in the smooth solvability problem of the following subsets of TE(2n), called implicitcit Hamiltonian systems.

1. Lagrangian submanifolds of TE(2n).
2. The image of an isotropic mapping from another E(2n) into TE(2n).
3. Generalized Hamiltonian systems over subvarieties of E(2n), introduced by P. A. M. Dirac.

1. LAGRANGIAN SUBMANIFOLDS OF THE TANGENT BUNDLE OF A SYMPLECTIC MANIFOLD AS IMPLICIT DIFFERENTIAL EQUATIONS

In the first lecture, we begin with observations on submanifolds of TE(n) as implicit differential equations. We will see that solvability of implicit differential equations is not so simple and we will encounter many pathetic phenomena. Here Theory of Singularities of Smooth mappings plays an important role. We will also see that smooth solvabilty of a submanifold as an implicit differential equation is strongly related with existence of a smooth solution of linear equation with parameter.

Then we investigate solvability of Lagrangian submanifolds of TE(2n) as implicit Hamiltonian systems. A submanifold of TE(2n) is called a Lagrangean submanifold of TE(2n) if the restriction of the symplectic structure of TE(2n) to the submanifold vanishes. Classical Hamiltonian vector fields are typical examples of Lagrangian submanifolds. Thus the notion of Lagrangian submanifolds as implicit Hamiltonian systems is a generalization of Hamiltonian dynamical systems and is important in mechanics.

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2. LIE ALGEBRA OF SOLVABLE ISOTROPIC MAPPINGS.

A smooth mapping F from a 2n dimensional manifold M into TE(2n) is called an isotropic mapping if the induced 2-form from the symplectic structure by F is null. The inclusion map of a Lagrangian submanifold of TE(2n) into TE(2n) is an isotropic mapping. Thus the notion of isotropic mappings is a generalization of Lagranian submanifolds. The image of isotropic mapping may have singularities while Lagranian submanifolds have no singularities. We investigate isotropic mappings regarding them as implicit diffferential equations.
Now consider a smooth mapping G from E(2n) into E(2n) and an isotropic mapping F such that the composition of F with the tangent bundle projection of TE(2n) coincides with G. In this case we may regard F as a vector field along G and let us call F an isotropic mapping along G. For a generic G, we will see that the set of smoothly integrable isotropic mappings along G has a structure of Lie algebra (Poisson algebra) with respect to the induced 2-form from the Darboux form by G. We will investigate this Lie algebra and classify Lie algebras which are stable in the sense of Singularity Theory of Smooth mappings.

3. SOLVABILITY OF GENERALIZED HAMILTONIAN SYSTEMS I, II.

Now let K be a submanifold of the base space E(2n) of the tangent bundle TE(2n) and consider Lagrangian submanifolds L of TE(2n) over K or images L of isotropic mappings over K, i.e. whose image under the tangent bundle projection is K. Since L is over K, trajectories of solutions of the implicit differential equation L stay within K, i.e., its dynamics is constrained. We call such L a generalized Hamiltonian system over a submanifold. We investigate the solvability problem of generalized Hamiltonian systems over submanifolds. We consider also the same problem when K is a singular subvariety of E(2n) and we obtain a condition for a generalized Hamiltonian system over a variety to be smoothly solvable.

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